On Anosov energy levels that are of contact type

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Abstract

In this work we prove that given an autonomous Lagrangian $L$ on a closed manifold $M$, if an Anosov energy level $k$ can be reparametrized to make it of contact type, then $k > c_0(L)$, the critical value of $L$ associated with the abelian covering.

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1 Introduction

Let $M$ be a closed connected manifold, $TM$ its tangent bundle. An autonomous Lagrangian is a smooth function, $L : TM \to \mathbb{R}$ convex and superlinear. This means that $L$ restricted to each $T_xM$ has positive definite Hessian and for some Riemannian metric we have

$$\lim_{|v| \to \infty} \frac{L(x, v)}{|v|} = \infty,$$

uniformly on $x$. Since $M$ is compact, the Euler-Lagrange equation defines a complete flow $\varphi_t$ on $TM$. Recall that the energy $E : TM \to \mathbb{R}$ is defined by

$$E(x, v) := \frac{\partial L}{\partial v}(x, v)v - L(x, v).$$

Since $L$ is autonomous, $E$ is a first integral of the flow $\varphi_t$. Let us set

$$e := \max_{x \in M} E(x, 0) = -\min_{x \in M} L(x, 0).$$

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Note that the energy level $E^{-1}\{k\}$ projects onto the manifold $M$ if and only if $k \geq e$.

We shall denote by $\mathcal{L}:TM \to T^*M$ the Legendre transform which is defined by $(x,v) \mapsto \frac{\partial L}{\partial v}(x,v)$. Our hypotheses on $L$ assure that $\mathcal{L}$ is a diffeomorphism. Let $H : T^*M \to \mathbb{R}$ be the Hamiltonian associated to $L$:

$$H(x,v) := \max_{v \in T_xM} \{ pv - L(x,v) \}.$$ 

If $\theta$ denotes the canonical 1-form on $T^*M$, then the Euler-Lagrange flow of $L$ can be also obtained as the Hamiltonian flow of $E$ with respect to the symplectic form on $TM$ given by $-\mathcal{L}^*d\theta$ thus, if $X$ denotes the vector field associated with the Euler-Lagrange flow then $i_X \mathcal{L}^*d\theta = -dE$. In other words, the energy function satisfies $E = H \circ \mathcal{L}$, so that energy levels for $L$ are sent to level sets of $H$.

**Definition:** An energy level $\Sigma = H^{-1}\{k\}$ is of **contact type** if there exists a 1-form $\lambda$ on $\Sigma$ such that $d\lambda = \omega (= -d\theta)$ and $\lambda(X) \neq 0$ on every point of $\Sigma$.

An Anosov energy level, is a regular energy level $E^{-1}\{k\}$ on which the flow $\varphi_t$ is an Anosov flow.

In [6] G. Paternain shows that if an Anosov energy level $k$ on a surface can be reparametrized to make it of contact type then $k > c_0(L)$ the critical value of $L$ associated with the abelian covering. Our goal in this note is to generalize this result, we shall prove the following:

**Theorem A.** Given an autonomous Lagrangian $L$ on a closed manifold $M$ with $\dim M \geq 2$, If an Anosov energy level $k$ can be reparametrized to make it of contact type then $k > c_0(L)$.

This completes the previous result.

### 2 Preliminaries and proofs

Let $\mathcal{M}(L)$ be the set of probabilities on the Borel $\sigma$-algebra on $TM$ that have compact support and are invariant under the flow $\varphi_t$. Let $H_1(M, \mathbb{R})$ be the first real homology group of $M$. Given a closed one-form $\omega$ on $M$ and $\rho \in H_1(M, \mathbb{R})$, let $\langle [w], \rho \rangle$ denote the integral of $\omega$ on any closed curve in the homology class $\rho$. If $\mu \in \mathcal{M}(L)$, its homology is defined as the unique $\rho(\mu) \in H_1(M, \mathbb{R})$ such that

$$\langle [w], \rho(\mu) \rangle = \int_M \omega d\mu,$$
for all closed 1-form on $M$. The integral on the right-hand side is with respect to $\mu$ with $\omega$ considered as the function $\omega : TM \to \mathbb{R}$.

Let $\mu$ be a $\varphi_t$-invariant probability supported on the energy level $\Sigma = E^{-1}\{k\}$. The Schwartzman’s asymptotic cycle $S(\mu) \in H_1(\Sigma, \mathbb{R})$ of $\mu$ is defined by

$$\langle [\Omega], S(\mu) \rangle = \int_{\Sigma} \Omega(X)d\mu,$$

for every closed 1-form $\Omega$ on $\Sigma$, where $X$ is the Lagrangian field on $\Sigma$.

The homology $\rho(\mu)$ of the measure $\mu$ is the projection of its asymptotic cycle by $\pi^* : H_1(\Sigma, \mathbb{R}) \to H_1(M, \mathbb{R})$.

Recall that the $L$-action of an absolutely continuous curve $\gamma : [a, b] \to M$ is defined by

$$AL(\gamma) := \int_a^b L(\gamma(t), \dot{\gamma}(t))dt.$$

Given two points $x_1, x_2 \in M$ and some $T > 0$ denoted by $C(x_1, x_2)$ the set of absolutely continuous curves $\gamma : [a, b] \to M$ with $\gamma(0) = x_1$ and $\gamma(T) = x_2$. For each $k \in \mathbb{R}$, we define

$$\Phi_k(x_1, x_2; T) := \inf\{A_{L+k}(\gamma) \mid \gamma \in C(x_1, x_2)\}.$$

The action potential $\Phi_k : M \times M \to \mathbb{R} \cup \infty$ of $L$ is defined by

$$\Phi_k(x_1, x_2) := \inf_{T>0} \Phi_k(x_1, x_2; T).$$

**Definition (Mañé):** The critical value of $L$ is the real number

$$c = c(L) := \inf\{k \mid \Phi_k(x, x) > -\infty \text{ for some } x \in M\}.$$

Note that if $k > c(L)$ actually $\Phi_k(x, x) > -\infty$ for all $x \in M$. Since $L$ is convex and superlinear, and $M$ is compact, such a number exists.

We can also consider the critical value of the lift of the Lagrangian $L$ to a covering of the compact manifold $M$. Suppose that $p : N \to M$ is a covering space and consider the Lagrangian $L : TN \to \mathbb{R}$ given by $L := Ldp$, for each $k \in \mathbb{R}$ we can define an action potential $\Phi_k$ in $N \times N$ just as above and similarly we obtain a critical value $c(L)$ for $L$. It can be easily checked that if $N_1$ and $N_2$ are coverings of $M$ such that $N_1$ covers $N_2$ then

$$c(L_1) \leq c(L_2),$$

where $L_1$ and $L_2$ denote the lifts of the Lagrangian $L$ to $N_1$ and $N_2$ respectively.
Among all possible coverings of $M$ there are two distinguished ones; the universal covering which we shall denote by $\widetilde{M}$, and the abelian covering which we shall denote by $\overline{M}$. The latter is defined as the covering of $M$ whose fundamental group is the kernel of the Hurewicz homomorphism $\pi_1(M) \to H_1(M,\mathbb{R})$; these coverings give rise to the critical values

$$c_u(L) := c(\widetilde{L}) \quad \text{and} \quad c_a(L) = c_0(L) := c(L)$$

where $\widetilde{L}$ and $L$ denote the lifts of the Lagrangian $L$ to $\widetilde{M}$ and $\overline{M}$ respectively. Therefore we have $c_u(L) \leq c_0(L)$, but in general the inequality may be strict as it was shown in [5].

2.1 Contact and Anosov energy levels

We begin by introducing some concepts related to Euler-Lagrange flow restricted on energy levels.

**Definition**: An energy level $\Sigma = H^{-1}\{k\}$ is of contact type if there exists a 1-form $\lambda$ on $\Sigma$ such that $d\lambda = \omega = -d\theta$ and $\lambda(X) \neq 0$ on every point of $\Sigma$.

Equivalently, if there exists a vector field $Y$ based on $\Sigma$, such that the Lie derivative $L_Y \omega = \omega$. The correspondence is given by $i_Y \omega = \lambda$.

The vector field $Y$ must be transverse to $\Sigma$ because if it is tangent to $\Sigma$ one has that $\lambda(X) = \omega(Y, X) = dH(Y) = 0$.

**Lemma 2.2.1** The set $\{k \in \mathbb{R} \mid H^{-1}\{k\} \text{ is of contact type} \}$ is open.

**Proof**: Suppose that $\Sigma = H^{-1}\{k\}$ is of contact type, then $k$ is a regular point of $H$, for otherwise the Hamiltonian flow contains a singularity on $\Sigma$ and that violates the condition $\lambda(X) \neq 0$. If $\lambda$ is a contact form for $\Sigma$, since $d\lambda = \omega$ then $\lambda = pdx|_{\Sigma} + \tau$, where $\tau$ is a closed 1-form on $\Sigma$. We can extend $\lambda$ as follows. Let $\pi : U \to \Sigma$ be the projection of an open neighbourhood $U$ of $\Sigma$ onto $\Sigma$. Let $\overline{\lambda} := pdx + \pi^*(\tau)$ then $d\overline{\lambda} = \omega$ and for $m$ near $k$ $d\overline{\lambda}|_{H^{-1}(m)} \neq 0$. □

The following criterion for contact type appears in [2]

**Proposition 2.2.2** If $L$ is a convex Lagrangian then an energy level $E^{-1}\{k\}$ is of contact type if and only if $\int_{TM}(L+k)d\mu > 0$ for any invariant measure $\mu$ supported $E^{-1}\{k\}$ with zero asymptotic cycle $\mathcal{S}(\mu) = 0$.

We shall need the following result:
Lemma 2.2.3 Suppose $M$ a closed connected manifold with $\dim M \geq 2$ and $M \neq T^2$. If $k > e$ then $\pi_* : H_1(E^{-1}\{k\}, \mathbb{R}) \to H_1(M, \mathbb{R})$ is an isomorphism.

Proof: Since $k > e$ and $\dim M \geq 2$ then the energy level $E^{-1}\{k\}$ is isomorphic to the unit tangent bundle of $M$ with the canonical projection. Using the Gysin exact sequence of the circle bundle $\pi : E^{-1}\{k\} \to M$ one can show that (see [3], lemma 1.45) the lemma follows if $M$ is orientable.

If $M$ is not orientable and $\dim M \geq 3$, using the exact homotopy sequence of the circle bundle $\pi : E^{-1}\{k\} \to M$:

$$0 = \pi_1(S^{n-1}) \to \pi_1(E^{-1}\{k\}) \xrightarrow{\pi_*} \pi_1(M) \to \pi_0(S^{n-1}) = 0,$$

thus we obtain that $\pi_* : \pi_1(E^{-1}\{k\}) \to \pi_1(M)$ is an isomorphism, which in turn implies that $\pi_* : H_1(E^{-1}\{k\}, \mathbb{R}) \to H_1(M, \mathbb{R})$ is a isomorphism. In the case that $M$ is not orientable and $\dim M = 2$, the proof is a minor modification of the above arguments. □

An Anosov energy level, is a regular energy level $E^{-1}\{k\}$ on which the flow $\varphi_t$ is an Anosov flow. In [1] was shown

Proposition 2.2.4 If the energy level $k$ is Anosov, then

$$k > c_u(L).$$

In [5] G. Paternain and M. Paternain gave examples of Anosov energy levels $k$ with $k < c_0(L)$ on surface of genus greater or equal than two. These examples give a negative answer to a question raised by Mane.

2.3 Proof of theorem A

Now we shall prove the theorem A, for this we use the next result of Paternain [4] and following his ideas we shall prove this result

Proposition 2.3.1 If $c_u(L) < k < c_0(L)$, there exists an invariant measure $\mu$ supported in the energy level $k$, such that $\rho(\mu) = 0$ and

$$\int_{E^{-1}\{k\}} (L + k)d\mu \leq 0.$$
Proof of theorem A: It follows from a result of Margulis that the energy levels of $T^2$ does not support Anosov flows thus in the case of $T^2$ the theorem is valid trivially. Therefore we can assume that $M \neq T^2$. Now as the flow is Anosov, by the proposition 2.2.4 we have that $k > c_u(L)$. But if the energy level $k \in (c_u(L), c_0(L))$ then applying the proposition 2.3.1, there exists an invariant measure $\mu$ such that $\rho(\mu) = 0$ and

$$\int_{E^{-1}(k)} (L + k)d\mu \leq 0.$$ 

therefore the lemma 2.2.3 and proposition 2.2.2 implies that, the energy level $k$ is not of contact type. Finally by lemma 2.2.1 the energy $k = c_0(L)$ cannot be of contact type then, we must have that $k > c_0(L)$. □

Proof of proposition 2.3.1: Since $k < c_0(L) = c_u(L)$ there exists $T > 0$ and an absolutely continuous closed curve $\gamma : [0, T] \to M$ homologous to zero such that

$$A_{L+k}(\gamma) < 0.\tag{1}$$

For $n \geq 1$, let us denote by $\gamma^n : [0, nT] \to M$ the curve $\gamma$ wrapped up $n$ times. Since $k > c_u(L)$, by (1) $\gamma^n$ cannot be homotopic to zero. Let $p : \tilde{M} \to M$ the covering projection and take $y$ such that $p(y) = \gamma(0) = \gamma(T)$, and let $\tilde{\gamma}^n : [0, nT] \to \tilde{M}$ be the unique lift of $\gamma^n$ with $\tilde{\gamma}^n(0) = y$. As $k > c_u(L)$ for each $n$ there exists a solution $x_n(t)$ of Euler-Lagrange with energy $k$ and some $T_n > 0$ such that $x_n(0) = y$ and $x_n(T_n) = \tilde{\gamma}^n(nT)$.

Let $\mu_n$ denote the probability measure in $TM$ uniformly distributed along $p \circ x_n|_{[0,T_n]}$ and take $\mu$ a point of accumulation of $\mu_n$, this measure $\mu$ has the required properties of the proposition 2.3.1. □

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References


