An upper bound on the size of irreducible quadrangulations *

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Abstract

Let $S$ be a closed surface with Euler genus $\gamma(S)$. A quadrangulation $G$ of a closed surface $S$ is irreducible if it does not have any contractible face. Nakamoto and Ota gave a linear upper bound for the number $n$ of vertices of $G$ in terms of $\gamma(S)$. By extending Nakamoto and Ota’s method we improve their bound to $n \leq 159.5\gamma(S) - 46$ for any closed surface $S$.

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1 Introduction

The orientable closed surface $M_g$ with genus $g$ is the sphere with $g$ handles attached. The non-orientable closed surface $N_g$ with genus $g$ is the sphere with $g$ cross-caps attached. The Euler genus of these surfaces is $\gamma(M_g) = 2g$ for the orientable surface $M_g$ and $\gamma(N_g) = g$ for the non-orientable surface $N_g$.

Let $G$ be a simple graph, that is, a graph without loops or parallel edges. The orientable genus $\bar{\gamma}(G)$ of $G$ is defined as the least $g$ such

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that $G$ is embeddable in $M_g$ and the non-orientable genus $\tilde{\gamma}(G)$ of $G$ is defined as the least $g$ such that $G$ is embeddable in $N_g$. The Euler genus $\gamma(G)$ of $G$ is defined to be $\gamma(G) = \min\{2\tilde{\gamma}(G), \tilde{\gamma}(G)\}$. Note that $\gamma(G) = \min\{\gamma(S) : G \text{ is embeddable in } S\}$.

A quadrangulation $G$ of a closed surface $S$ is a graph 2-cell embedded on $S$ in such a way that all faces of $G$ are quadrangles.

Let $G$ be a quadrangulation of a closed surface $S$ and let $abcd$ be a face of $G$. We say that $abcd$ is contractible if we can obtain a new quadrangulation by identifying $a$ with $c$ and deleting edges $ab$ and $cd$, see Figure 1. A quadrangulation $G$ is said to be irreducible if $G$ has no contractible face. The size of an irreducible quadrangulation can be measured in terms of its number of vertices, edges, or faces. By Euler’s formula these are all equivalent and we have chosen to measure the size in terms of the number $n$ of vertices.

Nakamoto and Ota [3] gave a linear upper bound for the number $n$ of vertices of an irreducible quadrangulation $G$ in terms of $\gamma(S)$, namely $n \leq 186\gamma(S) - 64$.

In this paper we prove an upper bound of $n \leq 159.5\gamma(S) - 46$ for the size of our quadrangulations by extending Nakamoto and Ota’s method.

2 Preliminaries

We use the following bound on the Euler genus of a 1- or 2-sum of graphs.

Lemma 2.1 (Miller [1]). Let $G_1$ and $G_2$ be two graphs and let $G := G_1 \cup G_2$. If $G_1$ and $G_2$ have at most two common vertices, then $\gamma(G) \geq \gamma(G_1) + \gamma(G_2)$.
From now on, let $S$ be either $M_g$ or $N_g$ with $g \geq 1$ and let $G$ be an irreducible quadrangulation of $S$. Let $G'$ be the graph embedded on $S$ obtained from $G$ by adding a vertex of degree four into each of its faces and joining it to the vertices of the corresponding face, see Figure 2.

Figure 2: Construction of $G'$.

For $v \in V(G)$ let $H_v$ be the subgraph of $G'$ induced by $v$, the vertices of the incident faces to $v$ in $G$ and the vertices added to the incident faces to $v$. Let $N_G(v)$ be the set of adjacent vertices to $v$ in $G$.

Nakamoto and Ota [2] proved the following result for $\deg_G(v) \leq 4$.

**Lemma 2.2.** Let $G$ be an irreducible quadrangulation of a closed surface $S$ and let $v$ be a vertex of $G$. Then $\gamma(H_v) \geq 1$.

**Proof.** Since $G$ is irreducible and $S$ is not the sphere, it follows that $G$ has no vertex of degree less than three. Let $v$ be a vertex of $G$ of degree $d \geq 3$ and let $W_v := v_0e_0v_1e_1 \ldots v_{2d-1}e_{2d-1}v_{2d}$ be a closed walk in $G$ such that $v_0, v_2, \ldots, v_{2d-2}$ are the neighbors of $v$ in clockwise direction and $v_2i v_{2i+1} v_{2i+2}$ is a face of $G$ for $i = 0, 1, \ldots, d - 1$. Since $G$ is irreducible every $v_{2i+1}$ must be adjacent or equal to some $v_j$ with $j \neq i, i+1$. We denote by $w_{2i+1}$ the new vertex of degree four added to the face $v_2i v_{2i+1} v_{2i+2}$ in $G$. Let $v_m, v_n \in W_d$, we define $\text{dist}(v_m, v_n) := \min\{|m - n| - 1, 2d - |m - n| - 1\}$, for $m \neq n$.

Let $v_\alpha, v_\beta \in W_d$ be vertices such that $v_\alpha \in N_G(v)$, $v_2i-1 v_\beta v_{2i+1} \in F(G)$, $v_\alpha$ is adjacent or equal to $v_\beta$ and $\text{dist}(v_\alpha, v_\beta) > 0$ is minimal. Without loss of generality we can assume that $\beta = 1$, $\alpha = 2k + 2$, and $\text{dist}(v_\alpha, v_\beta) = 2k$. 
Since $G$ is irreducible $v_{2k+1}$ is adjacent or equal to some neighbor of $v$, namely $v_{2j}$, with $k + 1 < j < d$ because $\text{dist}(v_\alpha, v_\beta) = 2k$ is minimal, see Figure 3.

Therefore we have a subdivision of $K_{3,3}$ with partition

$$\{v, w_{2k+1}, w_1\} \cup \{v_{2k+2}, v_{2k}, v_{2j}\}.$$ 

See Figure 4.

We say that a set $I$ of vertices of $G$ is face-independent if no two vertices in $I$ are incident to the same face of $G$. Nakamoto and Ota proved the following result for $k = 4$ [2].

**Lemma 2.3.** Let $G$ be an irreducible quadrangulation of a closed surface $S$ and let $k \geq 3$ be an integer. For each $i \geq 3$ let $V_i$ be the set of vertices
of degree \(i\) of \(G\). Then there exists an independent set \(X \subseteq V_3 \cup \cdots \cup V_k\) such that
\[
|X| \geq \sum_{i=3}^{k} \frac{|V_i|}{2i+1}.
\]

**Proof.** Let \(X_3\) be a maximal face-independent subset of \(V_3\). For \(i = 4, \ldots, k\) let \(X_i\) be a maximal face-independent subset from
\[
V_i - \bigcup_{j=3}^{i-1} A_{i,j}
\]
where \(A_{i,j}\) is the set of vertices of degree \(i\) that are incident to a face which is incident to a vertex in \(X_j\). We claim that \(X = \bigcup_{i=3}^{k} X_i\) satisfies the required property. Counting the vertices in the incident faces of each vertex \(x\) in \(X_i\) (including vertex \(x\)) we obtain
\[
(2i+1)|X_i| \geq |V_i| + \sum_{j=i+1}^{k} |A_{j,i}| - \sum_{j=3}^{i-1} |A_{i,j}| \quad \text{for every } 3 \leq i \leq k.
\]

Therefore
\[
|X| = \sum_{i=3}^{k} |X_i| \geq \sum_{i=3}^{k} \frac{|V_i|}{2i+1} + \sum_{i=3}^{k} \sum_{j=i+1}^{k} \frac{|A_{j,i}|}{2i+1} - \sum_{i=4}^{k} \sum_{j=3}^{i-1} \frac{|A_{i,j}|}{2i+1}.
\]

Observe that \(A_{i,j} = \emptyset\) for \(i \leq j\). Therefore
\[
\sum_{i=3}^{k} \sum_{j=i+1}^{k} \frac{|A_{j,i}|}{2i+1} - \sum_{i=4}^{k} \sum_{j=3}^{i-1} \frac{|A_{i,j}|}{2i+1} = \sum_{i=3}^{k} \sum_{j=i+1}^{k} \left[ \frac{|A_{j,i}|}{2i+1} - \frac{|A_{j,i}|}{2j+1} \right] \geq 0
\]
since \(j > i\). We conclude that
\[
|X| \geq \sum_{i=3}^{k} \frac{|V_i|}{2i+1}. \quad \square
\]

**3 Main Result**

**Theorem 3.1.** Let \(G\) be an irreducible quadrangulation of a closed surface \(S\) with \(n\) vertices. Then
\[
n \leq 159.5\gamma(S) - 46.
\]
Proof. Let \( m \) and \( f \) be the number of edges and faces of \( G \), respectively. By Euler’s formula
\[
 n - m + f = 2 - \gamma(S).
\]
Since \( G \) is a quadrangulation we have that \( 4f = 2m \) and therefore
\[
 4n - 2m = 8 - 4\gamma(S).
\]
Since \( \sum_{i \geq 3} |V_i| = n \) and \( \sum_{i \geq 3} i|V_i| = 2m \) we have that
\[
 3n + \sum_{i \geq 3} (1 - i)|V_i| = 8 - 4\gamma(S).
\]
Let \( k \geq 4 \) be an integer (to be chosen later). By adding and subtracting \( kn = k\sum_{i \geq 3} |V_i| \) we obtain
\[
 (3 - k)n + \sum_{i \geq 3} (k + 1 - i)|V_i| = 8 - 4\gamma(S).
\]
Thus
\[
 (1) \quad \sum_{i \geq 3} (k + 1 - i)|V_i| \geq (k - 3)n - 4\gamma(S) + 8.
\]
Let \( X \) be an independent set as in Lemma 2.3 and define \( Y := \{ y \in V(G) - X | y \in N_G(x) \text{ for some } x \in X \} \).

Consider the bipartite graph \( B \) with bipartition \( X \) and \( Y \), where \( xy \in E(B) \) for \( x \in X \), \( y \in Y \) if and only if \( xy \in E(G) \).

Let \( X' := \{ v_1, v_2, \ldots, v_r \} \) be a maximal subset of \( X \) satisfying the following condition:
\[
 \left| \bigcup_{1 \leq i < j} N_B(v_i) \cap N_B(v_j) \right| \leq 2, \text{ for each } j = 1, 2, \ldots, r.
\]
In other words, for each \( 2 \leq j \leq r \), \( v_j \) has at most two common neighbors with \( v_1, v_2, \ldots, v_{j-1} \). See Figure 5.

By Lemma 2.1 and Lemma 2.2 we obtain
\[
 \gamma \left( \bigcup_{i=1}^{r} H_{v_i} \right) \geq \sum_{i=1}^{r} \gamma(H_{v_i}) \geq r = |X'|.
\]
Since $\bigcup_{i=1}^{r} H_{v_i}$ is a subgraph of $G'$, it is embeddable in $S$, thus

$$\gamma(S) \geq \gamma \left( \bigcup_{i=1}^{r} H_{v_i} \right),$$

and it follows that

$$|X'| \leq \gamma(S).$$

Now define $Y' := \{ y \in Y | y \in N_B(v) \text{ for some } v \in X' \}$. Let $M$ be the subgraph of $B$ induced by $X \cup Y'$. Since $M$ is a subgraph of $G$ it is embeddable in $S$, therefore

$$|V(M)| - |E(M)| + |F(M)| \geq 2 - \gamma(S).$$

Since $M$ is bipartite each of its faces has at least 4 edges, therefore $4|F(M)| \leq 2|E(M)|$. Hence we have

$$|V(M)| - |E(M)| \geq 4 - 2\gamma(S).$$

By maximality of $X'$, each vertex $v \in X - X'$ has at least three neighbors in $Y'$. There are at least $|Y'|$ edges between $X'$ and $Y'$. Hence

$$|E(M)| \geq 3(|X| - |X'|) + |Y'|.$$

By replacing $|V(M)| = |X| + |Y'|$ and $|E(M)|$ in inequality (3), we obtain
\[4 - 2\gamma(S) \leq 2|X| + 2|Y'| - 3(|X| - |X'|) - |Y'| \]
\[\leq -|X| + |Y'| + 3|X'| \quad \text{since } |Y'| \leq 2k|X'| \]
\[\leq -\sum_{i=3}^{k} \frac{|V_i|}{2i+1} + (2k + 3)|X'| \quad \text{by Lemma 2.3.} \]

Let \( n_k \) be the smallest integer such that
\[
\frac{n_k}{2i+1} \geq k + 1 - i, \quad \text{for every } 3 \leq i \leq k.
\]
Since \((2i+1)(k - i + 1)\) has a unique maximum, we take this value as \( n_k \), namely
\[
n_k := \begin{cases} 
14 & \text{if } k = 4 \\
\frac{(k + 1)(k + 2)}{2} & \text{if } k \geq 5.
\end{cases}
\]

Thus we obtain
\[4 - 2\gamma(S) \leq -\frac{1}{n_k} \sum_{i=3}^{k} \frac{n_k}{2i+1} |V_i| + (2k + 3)|X'| \]
\[\leq -\frac{1}{n_k} [(k - 3)n + 8 - 4\gamma(S)] + (2k + 3)|X'| \]
and therefore
\[
(4) \quad \frac{(k - 3)n + 8 + 4n_k - (2k + 3)n_k|X'|}{4 + 2n_k} \leq \gamma(S).
\]

Thus, (2) provides a good bound for \( \gamma(S) \) when \( |X'| \) is large and (4) provides a good bound when \( |X'| \) is small. These two bounds are the same when their left-hand sides are equal, that is, when
\[
|X'| = \frac{(k - 3)n + 4n_k + 8}{(2k + 5)n_k + 4}.
\]
In particular, from (2) we obtain
\[
\frac{(k - 3)n + 4n_k + 8}{(2k + 5)n_k + 4} \leq \gamma(S),
\]
that is
\[ n \leq f(k)\gamma(S) - g(k), \]
where \( f(k) = \frac{(2k + 5)n_k + 4}{k - 3} \) and \( g(k) = \frac{4n_k + 8}{k - 3} \). A straightforward calculation shows that \( f(k) \) attains its minimum at \( k = 5 \), therefore
\[ n \leq 159.5\gamma(S) - 46. \]

Observe that for \( k = 4 \) we obtain \( n \leq 186\gamma(S) - 64 \), this is the bound obtained by Nakamoto and Ota [3].

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