On bounds for the stability number of graphs *

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Abstract

Let $G$ be a graph without isolated vertices and let $\alpha(G)$ be its stability number and $\tau(G)$ its covering number. In this paper we study the minimum number of edges a connected graph can have as a function of $\alpha(G)$ and $\tau(G)$. In particular we obtain the following lower bound:

$$q(G) \geq \alpha(G) - c(G) + \Gamma(\alpha(G), \tau(G)),$$

where $c(G)$ is the number of connected components of $G$ and

$$\Gamma(a, t) = \min \left\{ \sum_{i=1}^{a} \left( \frac{z_i}{2} \right) \right\} \bigg| \begin{array}{c}
    z_1 + \cdots + z_a = a + t \\
    \text{and } z_i \geq 0 \quad \forall i = 1, \ldots, a
\end{array},$$

for $a$ and $t$ two arbitrary natural numbers.

Also we prove that $\alpha(G) \leq \tau(G)[1 + \delta(G)]$, where $\delta(G) = \alpha(G) - \sigma_v(G)$ and $\sigma_v(G)$ is the $\sigma_v$-cover number of a graph, that is, the maximum natural number $m$ such that every vertex of $G$ belongs to a maximal independent set with at least $m$ vertices.

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1 Preliminaries

Let $G = (V, E)$ be a graph with $|V| = n$ vertices and $|E| = q$ edges. If $U \subseteq V$ is a subset of vertices, then the induced subgraph on $U$, denoted

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by $G[U]$, is the graph with $U$ as a vertex set and whose edges are precisely the edges of $G$ with both ends in $U$.

A subset $A \subset V$ is a minimal vertex cover for $G$ if: (i) every edge of $G$ is incident with at least one vertex in $A$, and (ii) there is no proper subset of $A$ with the first property. If $A$ satisfies condition (i) only, then $A$ is called a vertex cover of $G$.

The vertex covering number of $G$, denoted by $\tau(G)$, is the number of vertices in a minimum vertex cover in $G$, that is, the size of any smallest vertex cover in $G$.

It is convenient to regard the empty set as a minimal vertex cover for a graph with all its vertices isolated.

A subset $M$ of $V$ is called a stable set if no two vertices in $M$ are adjacent. We call $M$ a maximal stable set if it is maximal with respect to inclusion. The stability number of a graph $G$ is given by

$$\alpha(G) = \max\{|M| \mid M \subset V(G) \text{ is a stable set in } G\}.$$  

Note that a set of vertices in $G$ is maximal stable set if and only if its complement is a minimal vertex cover for $G$.

Thus we have $\alpha(G) + \tau(G) = n$.

A subset $W$ of $V$ is called a clique if any two vertices in $W$ are adjacent. We call $W$ maximal if it is maximal with respect to inclusion. The clique number of a graph $G$ is given by

$$\omega(G) = \max\{|W| \mid W \subset V(G) \text{ is a clique in } G\}.$$  

Given a subset $U \subset V$, the neighbour set of $U$, denoted by $N(U)$, is defined as $N(U) = \{v \in V \mid v \text{ is adjacent to some vertex in } U\}$.

### 2 The number of edges of a connected graph with fixed stability number

We give in Theorem 2.3 a lower bound for the number of edges of a graph $G$ as a function of the stability number $\alpha(G)$, the covering number $\tau(G)$ and the number of connected components $c(G)$ of $G$. This is an answer to an open question posed by Ore in his book [6] which is a variant for connected graphs of a celebrated theorem of Turan [7].

For a graph $G = (V, E)$, we will denote by $q(G)$ the cardinality of the edge set $E(G)$ of $G$. We say that a connected graph $G$ is $q$-minimal if there is no graph $G'$ such that
(i) $G'$ is connected,
(ii) $\alpha(G') = \alpha(G)$,
(iii) $\tau(G') = \tau(G)$ and
(iv) $q(G') < q(G)$.

Hence if $G$ is $q$-minimal, then either $\alpha(G) < \alpha(G - e)$ or $c(G) < c(G - e)$ for all the edges $e$ of $G$ (note that $\alpha(G) < \alpha(G - e)$ if and only if $\tau(G) > \tau(G - e)$). That is, an edge of a $q$-minimal graph is either $\alpha$-critical or a bridge. Therefore the blocks of a $q$-minimal graph are $\alpha$-critical graphs. Here an edge $e$ of a graph $G$ is $\alpha$-critical if $\alpha(G - e) = \alpha(G) + 1$. $G$ is $\alpha$-critical if all the edges of $G$ are $\alpha$-critical and is a $\tau$-critical graph if $\tau(G - v) = \tau(G) - 1$ for all the vertices $v$ of $G$.

In order to bound the number of edges we introduce the following numerical function. Let $a$ and $t$ be two natural numbers and let

$$\Gamma(a, t) = \min \left\{ \sum_{i=1}^{a} \left( \frac{z_i}{2} \right) \mid \begin{array}{l}
z_1 + \cdots + z_a = a + t \\
\text{and } z_i \geq 0 \quad \forall i = 1, \ldots, a
\end{array} \right\},$$

Lemma 2.1 Let $a$ and $t$ be natural numbers, then

(i) $\Gamma(a, t) = (a - s) \left( \frac{r}{2} \right) + s \left( \frac{r + 1}{2} \right)$ where $a + t = r(a) + s$ with $0 \leq s < a$.

(ii) $\Gamma(a - 1, t) - \Gamma(a, t) \geq \frac{1}{2} \left( \left| \frac{a+t}{a} \right|^2 - \left| \frac{a+t}{a} \right| \right) \geq 0$ for all $a \geq 2$ and $t \geq 1$.

Moreover we have that $\Gamma(a - 1, t) - \Gamma(a, t) = \frac{1}{2} \left( \left| \frac{a+t}{a} \right|^2 - \left| \frac{a+t}{a} \right| \right)$ if and only if $\left| \frac{a+t}{a} \right| \geq \frac{t}{a-1}$ and we have that $\left| \frac{a+t}{a} \right|^2 - \left| \frac{a+t}{a} \right| = 0$ if and only if $0 \leq t < a$.

(iii) $\Gamma(a, t) - \Gamma(a, t - 1) = 1 + \left[ \frac{t-1}{a} \right] = \left[ \frac{t}{a} \right]$ for all $a \geq 1$ and $t \geq 2$.

(iv) $\sum_{i=1}^{k} \Gamma(a_i, t_i) \geq \Gamma(\sum_{i=1}^{k} a_i, \sum_{i=1}^{k} t_i)$ for all $a_i \geq 1$ and $t_i \geq 1$.

Furthermore we have that

$$\Gamma(a_1, t_1) + \Gamma(a_2, t_2) = \Gamma(a_1 + a_2, t_1 + t_2)$$

if and only if $\left[ \frac{t_1}{a_1} \right] = \left[ \frac{t_2}{a_2} \right]$.

(v) $\left[ \frac{2(a-1+\Gamma(a, t))}{a+t} \right] = 1 + \left[ \frac{t}{a} \right] + L$, where $L = -1$ if and only if $a = 1$, $L = 0$ if and only if either $a \nmid t$ (a does not divide $t$) or $a \mid t$ (a divides $t$) and $a \geq 2$ and $L = 1$ if either $1 \leq t < a$ or $a + 2 \leq t < 2a$. 

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Proof: (i) The case for \( a = 1 \) is trivial. For \( a \geq 2 \) we will use the next result.

Claim 2.2 Let \( n, m \geq 1 \) be natural numbers with \( n > m + 1 \), then

\[
\binom{n}{2} + \binom{m}{2} > \binom{n-1}{2} + \binom{m+1}{2}.
\]

Proof: It follows easily, since \( \binom{n}{2} - \binom{n-1}{2} = n - 1 \).

Let \( a \geq 2 \) and \( t \geq 1 \) be fixed natural numbers, \((z_1, \ldots, z_a) \in \mathbb{N}^a\) such that \( \sum_{i=1}^a z_i = a + t \) and let \( L(z_1, \ldots, z_a) = \sum_{i=1}^a \binom{z_i}{2} \). Now, if

\[
\{z_1, \ldots, z_a\} \neq \{r, \ldots, r; \underbrace{r + 1, \ldots, r + 1}_s\}
\]

where \( a + t = r(a) + s \) with \( 0 \leq s < a \), then there exist \( z_{i_1} \) and \( z_{i_2} \) with \( z_{i_1} > z_{i_2} + 1 \). Applying Claim 2.2 we have that

\[
L(z_1, \ldots, z_a) > L(z_1, \ldots, z_{i_1} - 1, \ldots, z_{i_2} + 1, \ldots, z_a) \geq \Gamma(a, t),
\]

and therefore we obtain the result. \( \square \)

(ii) Let \( a + t = ar + s \) with \( r \geq 1 \) and \( 0 \leq s < a \), then

\[
a + t - 1 = (a - 1)(r + l) + s'
\]

where \( r + s - 1 = (a - 1)l + s' \) with \( l \geq 0 \) and \( 0 \leq s' < a - 1 \).

Using part (i) and after some algebraic manipulations we obtain that

\[
2(\Gamma(a-1, t) - \Gamma(a, t)) = (r^2 - r) + (l^2 - l)(a - 1) + 2ls'.
\]

Therefore \( \Gamma(a-1, t) - \Gamma(a, t) \geq \frac{1}{2} \left( \left\lfloor \frac{a+t}{a} \right\rfloor^2 - \left\lfloor \frac{a+t}{a} \right\rfloor \right) \geq 0 \), since \( r, l, s' \geq 0 \) and \( u^2 - u \geq 0 \) for all \( u \geq 0 \). Moreover we have that \( \Gamma(a-1, t) - \Gamma(a, t) = \frac{1}{2} \left( \left\lfloor \frac{a+t}{a} \right\rfloor^2 - \left\lfloor \frac{a+t}{a} \right\rfloor \right) \) if and only if

\[
(l, s') = \begin{cases} 
(0, s') & \\
(1, 0) & 
\end{cases}
\]

These two possibilities imply that \( r + s < a \) and \( r + s = a \), respectively. Finally it is clear that \( \left\lfloor \frac{a+t}{a} \right\rfloor^2 - \left\lfloor \frac{a+t}{a} \right\rfloor = 0 \) if and only if \( 0 \leq t < a \).
(iii) Let $a + t - 1 = ar + s$ with $r \geq 1$ and $0 \leq s < a$, then

$$a + t = \begin{cases} 
    ar + (s + 1) & \text{if } 0 \leq s < a - 1 \\
    a(r + 1) & \text{if } s = a - 1
\end{cases}$$

and by (i) we have that

$$\Gamma(a, t) - \Gamma(a, t - 1) =$$

$$= \left\{ \begin{array}{l}
    (a - s - 1) \left( \frac{r}{2} \right) + (s + 1) \left( \frac{r + 1}{2} \right) - (a - s) \left( \frac{r}{2} \right) + s \left( \frac{r + 1}{2} \right) \\
    a \left( \frac{r + 1}{2} \right) - \left( \frac{r}{2} \right) + (a - 1) \left( \frac{r + 1}{2} \right)
\end{array} \right.$$ 

$$= \frac{r + 1}{2} - \left( \frac{r}{2} \right) = r = \left\lfloor \frac{a + t - 1}{a} \right\rfloor.$$

(iv) Follows directly from the definition of $\Gamma(a, t)$.

(v) Let $a + t = ar + s$ with $r \geq 1$ and $0 \leq s < a$ then, by (i) we have that

$$\left\lfloor \frac{2(a - 1 + \Gamma(a, t))}{a + t} \right\rfloor = \left\lfloor \frac{2 \left( a - 1 + (a - s) \left( \frac{r}{2} \right) + s \left( \frac{r + 1}{2} \right) \right)}{a + t} \right\rfloor$$

$$= \left\lfloor \frac{2(a - 1) + (a - s)r(r - 1) + s(r + 1)r}{a + t} \right\rfloor$$

$$= \left\lfloor \frac{2(a - 1) + r(ar + s) - r(a - s)}{ar + s} \right\rfloor$$

$$= r + \left\lfloor \frac{2(a - 1) - r(a - s)}{ar + s} \right\rfloor$$

$$= \left\lfloor \frac{a + t}{a} \right\rfloor + \left\lfloor \frac{2(a - 1) - r(a - s)}{ar + s} \right\rfloor.$$
Finally

\[ L = \left\lceil \frac{2(a-1) - r(a-s)}{ar+s} \right\rceil \]

\[ = \begin{cases} 
-1 & \text{if } a = 1, \\
0 & \text{if } a \not| t \text{ or } a|t, \text{ and } a \geq 2, \\
1 & \text{if either } r = 1 \text{ and } s \geq 1 \text{ or } r = 2 \text{ and } s \geq 2.
\end{cases} \]

Since

\[ L \geq -1 \iff -2(ar+s) < r(s-a) + 2(a-1) \]
\[ \iff 2 < (a+s)(2+r) \]
\[ \iff a, r \geq 1, \]
\[ L \geq 0 \iff -(ar+s) < r(s-a) + 2(a-1) \]
\[ \iff 2 < s(r+1) + 2a \]
\[ \iff s > 0 \text{ or } s = 0 \text{ and } a \geq 2, \]
\[ L \geq 1 \iff 0 < r(s-a) + 2(a-1) \]
\[ \iff 0 < (a-1)(2-r) + r(s-1) \]
\[ \iff r = 1, s \geq 1 \text{ or } r = 2, s \geq 2. \]

**Theorem 2.3** ([4, Theorem 3.3]) Let \( G \) be a graph, then

\[ q(G) \geq \alpha(G) - c(G) + \Gamma(\alpha(G), \tau(G)). \]

**Proof:** We will use induction on \( \tau(G) \), the covering number of \( G \).

For \( \tau(G) = 1 \) it is easy to see that the unique connected graphs with \( \tau(G) = 1 \) are the stars \( K_{1,n} \) (\( \alpha(K_{1,n}) = n - 1 \)) and the result follows, since

\[ q(K_{1,n}) = n - 1 = (n - 1) - 1 + 1 = \alpha(K_{1,n}) + c(K_{1,n}) + \Gamma(n-1,1). \]

In the same way it is easy to see that the unique graphs \( G \) with \( \alpha(G) = 1 \) are the complete graphs \( K_n \) (\( \tau(K_n) = n - 1 \)). Since we have that,

\[ q(K_n) = \binom{n}{2} = 1 - 1 + \binom{n}{2} = \alpha(K_n) + c(K_n) + \Gamma(1,n-1), \]

it follows that the family of complete graphs satisfies the result.
Moreover the graphs of both families are $q$-minimal graphs. So we can assume that the result is true for $\tau(G) \leq k > 1$.

Let $G$ be a $q$-minimal graph with $\tau(G) = k + 1$. Since $q(G) = \sum_{i=1}^{s} q(G_i)$, $\alpha(G) = \sum_{i=1}^{s} \alpha(G_i)$ and $\tau(G) = \sum_{i=1}^{s} \tau(G_i)$ where $G_1, \ldots, G_s$ are the connected components of $G$, it follows from Lemma 2.1(iv) that we can assume with out loss of generality that $G$ is connected and $\alpha(G) \geq 2$.

Let $e$ be an edge of $G$ and consider the graph $G' = G - e$. We have two possibilities

\[ \tau(G') = \begin{cases} 
\tau(G) \\
\tau(G) - 1 
\end{cases} \]

That is, an edge of $G$ is either a bridge or $\alpha$-critical.

**Case 1** First assume that $G$ has no bridges, that is, $G$ is a $\alpha$-critical graph. Let $v$ be a vertex of $G$ of maximum degree. Since any $\alpha$-critical graph is $\tau$-critical we have that $\tau(G - v) = \tau(G) - 1$ and $\alpha(G - v) = \alpha(G)$, moreover since the $\alpha$-critical graphs are blocks we have that $G - v$ is connected. Now, by the induction hypothesis we have that

\[ q(G - v) \geq \alpha(G) - 1 + \Gamma(\alpha(G), \tau(G) - 1). \]

Using the formula

\[ \sum_{i=1}^{\alpha(G-v)+\tau(G-v)} \deg(v_i) = 2q(G - v) \]

we conclude that there must exist a vertex $v' \in V(G - v)$ with

\[ \deg(v') \geq \left\lceil \frac{2q(G - v)}{\alpha(G - v) + \tau(G - v)} \right\rceil \]

\[ \geq \left\lceil \frac{2(\alpha(G) - 1 + \Gamma(\alpha(G), \tau(G) - 1))}{\alpha(G) + \tau(G) - 1} \right\rceil. \]

Now by Lemma 2.1(iii) and (v) we have that

\[ q(G) = q(G - v) + \deg(v) \]

\[ \geq \alpha(G) - 1 + \Gamma(\alpha(G), \tau(G) - 1) + \deg(v') \]

\[ \geq \alpha(G) - 1 + \Gamma(\alpha(G), \tau(G)). \]

So, if the graph $G$ has an edge that is a bridge, we have that $c(G') = c(G) - 1 = 2$. Denote by $G_1$ and $G_2$ the connected components of $G - e$. We have two more cases:
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Case 2 Assume that $\tau(G_1) > 0$ or $\tau(G_2) > 0$, then $\tau(G_1) \leq k$ and $\tau(G_2) \leq k$ and by the induction hypothesis we have that

$$q(G_1) \geq \alpha(G_1) - 1 + \Gamma(\alpha(G_1), \tau(G_1)),$$

$$q(G_2) \geq \alpha(G_2) - 1 + \Gamma(\alpha(G_2), \tau(G_2)).$$

Using the above formulas and Lemma 2.1(iv) we have that

$$q(G) = q(G_1) + q(G_2) + 1$$

$$\geq \alpha(G_1) - 1 + \alpha(G_2) - 1 + \Gamma(\alpha(G_1), \tau(G_1)) + \Gamma(\alpha(G_2), \tau(G_2)) + 1$$

$$(iv) \geq \alpha(G) - 1 + \Gamma(\alpha(G), \tau(G))$$

Note that $\alpha(G) = \alpha(G_1) + \alpha(G_2)$ and $\tau(G) = \tau(G_1) + \tau(G_2)$.

Case 3 Assume that there does not exist a bridge satisfying the above conditions, that is, for all the bridges of $G$ we have that $\tau(G_1) = 0$ or $\tau(G_2) = 0$. In this case we must have that $G$ is equal to an $\alpha$-critical graph $G_1$ with a vertex of $G_1$ being the center of a star $K_{1,l}$. Moreover we have that $\tau(G) = \tau(G_1)$ and $\alpha(G) = l + \alpha(G_1)$ because $G_1$ is vertex-critical and therefore each vertex belongs to a minimum vertex cover. Now using Case 1 and Lemma 2.1(ii), we obtain,

$$q(G) = l + q(G_1) \geq l + (\alpha(G_1) - 1 + \Gamma(\alpha(G_1), \tau(G_1)))$$

$$= \alpha(G) - 1 + \Gamma(\alpha(G_1), \tau(G))$$

$$(ii) \geq \alpha(G) - 1 + \Gamma(\alpha(G), \tau(G)).$$

3 A classification of $q$-minimal graphs

A 1-linking of a graph $G$ is a new graph $G'$ with the same vertex set as $G$ but obtained from $G$ by adding the minimum number of edges possible such that $G'$ be connected. The graph $G$ is called the subjacent graph of the 1-linking graph $G'$ and the edges that we add are called the linking edges.

Clearly a 1-linking graph $G'$ of a disconnected graph $G$ can be obtained by adding $e(G) - 1$ edges, where $e(G)$ is the number of connected components of $G$. This definition is equivalent to the one given in [1] of a tree-linking of a graph.
A graph $G$ is a Turán graph, denoted by $T(a, t)$, if $G$ is the disjoint union of $a - s$ complete graphs with $r$ vertices and $s$ complete graphs with $r + 1$ vertices, where $a + t = r(a) + s$ with $0 \leq s < a$.

A graph $G$ with covering number $\tau(G) = t$ and stability number $\alpha(G) = a$ is said to be a transformed Turán graph or TT graph if either $G$ is isomorphic to $T(a, t)$, or $a \leq t \leq 2a$ and $G$ can be obtained from $T(a, t)$ by the following construction:

Take a positive integer $k$ such that $k \leq \min\{k_2, k_3\}$ where $k_2, k_3$ denote the number of copies in $T(a, t)$ of $K_2$ and $K_3$ respectively. For every $1 \leq i \leq k$ replace $j_i$ copies of $K_2$ and one copy of $K_3$ by a cycle $C_{2j_i+3}$, where $j_1 + \cdots + j_k \leq k_2$.

Given a 1-linking $G'$ of $G$, we define a leaf in $G'$ as a connected component $G_i$ of $G$ incident to a unique linking edge or as a connected component $G_i$ with the property that there exist a unique vertex $v$ in $G_i$ such that all linking edges with one end in $G_i$ are incident to the vertex $v$.

**Lemma 3.1** A graph $G$ is $q$-minimal if and only if $G$ is a 1-linking of a transformed Turán graph.

Proof: We will use double induction on the stability and covering number of the graph. For $\alpha(G) = 1$ we have that $G$ must be a complete graph and the result is clear. Therefore we can assume that $G$ is a $q$-minimal graph with $\alpha(G) \geq 2$.

If $G$ is not 2-connected, using the same arguments used in cases 2 and 3 in the proof of Theorem 2.3 and the induction hypothesis the result follows readily. Hence we can assume that $G$ is a 2-connected graph, in fact that, $G$ is an $\alpha$-critical graph. Therefore the proof will be complete if we prove that $G$ is an odd cycle.

Let $G$ be a $q$-minimal and $\alpha$-critical graph and let $v \in V(G)$ be a vertex of maximal degree.

**Claim 3.2** $G \setminus v$ is $q$-minimal.

Proof: Assume that $G \setminus v$ is not $q$-minimal, then by the same arguments as those in case 1 in the proof of Theorem 2.3 and Lemma 2.1 we have

$$q(G) = q(G \setminus v) + \deg(v) \geq q(G \setminus v) + \deg(v') \geq \alpha(G) + \Gamma(\alpha(G), \tau(G) - 1) + \left[\frac{\alpha(G) + \tau(G) - 1}{\alpha(G)}\right] \tag{\text{iii}}$$
which is a contradiction to the $q$-minimality of $G$. \hfill \Box

Since $G$ is $\alpha$-critical and in particular $\tau$-critical, then $\alpha(G) = \alpha(G \setminus v)$, that is, the set of vertices $N(v)$ must satisfy that $N(v) \cap M \neq \emptyset$ for all maximum stable sets $M$ of $G \setminus v$. Hence $\alpha(G \setminus N[v]) = \alpha(G) - 1$, where $N[v] = N(v) \cup \{v\}$. A set of vertices $N$ in $G \setminus v$ can be the set of neighbors of $v$ in $G$ if and only if $V(G \setminus v) \setminus N$ induce a subgraph $G'$ of $G \setminus v$ with $\alpha(G') = \alpha(G) - 1$. Moreover $N$ is minimal under inclusion if and only if $G[V(G \setminus v) \setminus N]$ is maximal under inclusion.

Now, since $G \setminus v$ is $q$-minimal, by induction hypothesis $G \setminus v$ is a 1-linking of a TT graph. In this case it is easy to find the maximal induced subgraph $G'$ of $G \setminus v$ with $\alpha(G') = \alpha(G) - 1$.

Claim 3.3 Let $H$ be a TT graph with $H_1, \ldots, H_a$ connected components and let $L$ be a 1-linking of $H$. Take $L'$ to be a maximal induced subgraph of $L$ with $\alpha(L') = \alpha(L) - 1$, then we have that

(i) $L'$ is induced by the set of vertices $V(L) \setminus V(H_i)$, for some $H_i$ with $\alpha(H_i) = 1$, or

(ii) $L'$ is induced by the set of vertices in $V(L) \setminus \{v_1, v_2, v_3\}$, where $\{v_1, v_2, v_3\}$ are vertices of an odd cycle $H_j$ such that $H_j \setminus \{v_1, v_2, v_3\}$ is a disjoint union of paths with an even number of vertices, or

(iii) $L'$ satisfies the following conditions: (1) $V(H_i) \cap V(L') \neq \emptyset$ for all $H_i$, (2) if $H_i$ is an odd cycle, then $V(H_i) \subset V(L')$, (3) if $H_i$ is a complete graph such that $V(H_i) \not\subset V(L')$, then for all $v \in V(H_i) \cap V(L')$ there exist at least one linking edge $e_v$ incident to $v$.

Proof: If $V(L') \cap V(H_i) = \emptyset$ for some $1 \leq i \leq a$ with $\alpha(H_i) = 1$, then $L' = L[V(H) \setminus V(H_i)]$, since $V(L') \subseteq V(H) \setminus V(H_i)$ and $\alpha(L[V(H) \setminus V(H_i)]) = \alpha(L) - 1$.

Therefore we can assume that if $L' \neq L \setminus H_i$ with $\alpha(H_i) = 1$, then $V(H_i) \cap V(L') \neq \emptyset$ for all $H_i$ with $\alpha(H_i) = 1$.

If $H_j$ is an odd cycle with $2m + 1$ vertices and since all the proper induced graphs of a cycle are paths $P_n$ with $\alpha(P_n) = \lfloor \frac{n}{2} \rfloor$, then $\alpha(H_j \setminus C) = \alpha(H_j) - 1$ for some $C \subset V(H_j)$ if and only if $H_j[C^c]$ is a disjoint union of three paths $P_{m_1}, P_{m_2}, P_{m_3}$ with $m_1, m_2, m_3 \geq 0$ even numbers and such that $m_1 + m_2 + m_3 = 2(m - 1)$. Therefore, either $L'$ is described as in (ii) or $V(H_j) \subset V(L')$ for all $H_j$ with $\alpha(H_j) \geq 2$.

To finish, if $L'$ is not given by (i) or (ii), then we can assume that $V(L') \cap V(H_i) \neq \emptyset$ for all $1 \leq i \leq a$, moreover if $H_j$ is an odd cycle then
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\( V(H_j) \subset V(L') \). Clearly if \( v \in V(H_i) \cap V(L') \) and \( v \) is not incident to any linking edge, then \( V(H_i) \subset V(L') \) because \( \alpha(L') = \alpha(L[V(L') \cup V(H_i)]) \).

\( \square \)

Applying Claim 3.3 to \( G \setminus v \) it is easy to conclude that

- \( G \) is a complete graph whenever \( G \setminus N[v] \) is as in (i).
- \( G \) is not \( q \)-minimal whenever \( G \setminus N[v] \) is as in (ii).

Therefore it only remains to consider when \( G \setminus N[v] \) satisfies the conditions given in case (iii). Let \( H_{i_0} \) be a complete graph such that \( H_{i_0} \) is a connected component of the subjacent graph of \( G \setminus v \) (a TT graph) with \( V(H_{i_0}) \not\subset V(G') \) (note that by Claim 3.3 (iii) there exists at least one graph \( H_i \) with this condition) and take \( P = V(H_{i_0}) \setminus P \) and \( Q = V(H_{i_0}) \setminus P \). Since \( G \setminus v \) is \( q \)-minimal, then we have that for all \( u \in P \), \( (G \setminus v) \setminus u \) in not connected. For all \( u \in P \), let \( G_u \) the union of the connected components of \( (G \setminus v) \setminus u \) such that \( V(G_u) \cap V(H_{i_0}) = \emptyset \). Note that \( G_u \) is an induced subgraph (a disjoint union of 1-linking of TT graphs) of \( G \setminus v \) such that \( G_u \) is joined to \( u \) by linking edges. Note that if \( |V(H_{i_0})| \geq 2 \), then \( G_u \) is unique.

Here we need to consider two cases, the first case is when \( G_u \) is not a TT graph. If \( S \) is a leaf of \( G_u \) not joined to \( u \) by a linking edge, then by the 2-connectivity of \( G \) we have that \( v \) must be incident with at least one vertex of \( S \). In the other case, if \( G_u \) is a TT graph, then by the 2-connectivity of \( G \) we have that there exists at least one vertex \( w \) such that \( w \) is incident to \( v \) and we can consider that \( S = G_u \) is the only leaf of \( G_u \).

Moreover, by Claim 3.3 (iii) we have that if \( v_s \) is the unique vertex of \( S \) such that all the linking edges with one end in \( S \) are incident to \( v_s \), then \( v \) must be incident with all the vertices of \( S \setminus v_s \), more precisely
we have that:

\[(3) \quad \deg(v) \geq |Q| + \sum_{u \in P} \sum_{H_j \in L(G_u)} (|H_j| - 1) \geq |H_{i_0}|,\]

where \(L(G_u)\) is either the set of leaves of \(G_u\) not joined to \(u\) when \(G_u\) is not a TT graph or equal to \(G_u\) when \(G_u\) is a TT graph. We have equality in \((*)\) if and only if all the leaves of \(G_u\) are isomorphic to \(K_2\) and if \(G_u\) is not a TT graph, then \(G_u\) has exactly two leaves.

Now, let \(H_{i_0}\) with \(|H_{i_0}| = k = \left\lfloor \frac{|V(G \setminus v)|}{\alpha(G)} \right\rfloor\), using that

\[
\deg(v) = q(G) - q(G \setminus v) \\
\leq \Gamma(\alpha(G), \tau(G)) - \Gamma(\alpha(G), \tau(G) - 1) \\
= \left\lfloor \frac{|V(G)|}{\alpha(G)} \right\rfloor - 1 \\
\leq k
\]

and \(\deg(v) \geq 2k - 2\) (by Equation (3)), then \(k\) must be equal to 2. A similar argument shows that \(k = 2\) when we take \(|H_{i_0}| = k = \left\lfloor \frac{|V(G \setminus v)|}{\alpha(G)} \right\rfloor\).

Hence \(H_{i_0} = K_2\), \(\deg(v) = 2\), \(\left\lfloor \frac{|V(G \setminus v)|}{\alpha(G)} \right\rfloor \leq 3\), \(|P| = 1\) and \(G_u\) has only two leaves. Therefore \(G\) must be an odd cycle since \(G_u\) must be a 1 - linking of a TT graph whose components are all isomorphic to \(K_2\).

\[\square\]

4 A relation between the stability and covering number

In this section we present some relations between two important invariants of a graph \(G\), the stability number \(\alpha(G)\) and covering number \(\tau(G)\).

The origin of our interest in the study of these relations comes from monomial algebras, more precisely we have that: the stability number \(\alpha(G)\) of a graph \(G\), is equal to the dimension of the Stanley-Reisner ring associated to the graph \(G\); and the covering number \(\tau(G)\) of \(G\) is equal to the height of the ideal associated to the graph \(G\). Finally, \(\alpha(G) - \delta(G)\) is an upper bound to the depth of this ring.
From the algebraic point of view an important class of rings is given by those rings $R$ such that their dimension is equal to their depth. The rings in this class are called Cohen-Macaulay rings.

A graph is Cohen-Macaulay if the Stanley-Reisner ring associated to it, is Cohen-Macaulay. We have that if a graph $G$ is Cohen-Macaulay, then $\delta(G) = 0$; note that this is a necessary condition but not a sufficient condition.

The family of graphs with $\delta(G) \geq 1$ correspond to the Stanley-Reisner rings that have a large depth. Moreover, the dimension minus the depth is bounded below by $\delta(G)$, and hence $\delta(G)$ is a measure of how far these rings are from being Cohen-Macaulay.

The following results are in the spirit of [3], in that paper the authors where motivated in bounding invariants for edge rings. In this paper we concentrate mainly in the combinatorial aspects of these bounds.

The theorem below, gives an idea of the class of graphs that are Cohen-Macaulay and of those graphs that are far from being Cohen-Macaulay. We thank N. Alon (private communication) for some useful suggestions in making the proof of this result simpler and more readable.

**Theorem 4.1** Let $G$ be a graph without isolated vertices, then

$$\alpha(G) \leq \tau(G)[1 + \delta(G)].$$

**Proof:** First, fix a minimal vertex cover $C$ with $\tau(G)$ vertices. Let $v \in C$, then there exist a maximal stable set $M'$ with $v \in M'$ and $|M'| \geq \sigma_v(G)$. Hence there exist a natural number $k \leq \tau(G)$ and $T_1, \ldots, T_k$ maximal stable sets with $|T_i| \geq \sigma_v(G)$ such that

$$C \subset \bigcup_{i=1}^{k} T_i.$$

Let $M = V \setminus C$ and take $C_i = C \cap T_i$ and $M_i = M \cap T_i$ for all $i = 1, \ldots, k$. Since the graph $G$ does not have isolated vertices, then for all $v \in M$ there exists an edge $e$ of $G$ with $e = \{v, v'\}$. Now, as $C = V(G) \setminus M$ and $C$ is a vertex cover we have that $v' \in C$, that is

$$M = \bigcup_{i=1}^{k} (M \cap N(C_i)).$$
Since $S_i = V(G) \setminus T_i = (C \setminus C_i) \cup (M \setminus M_i)$ is a minimal vertex cover with $|S_i| \leq n - \sigma_v(G)$ for all $i = 1, \ldots, k$, then
\[ |C \setminus C_i| + |M \setminus M_i| = |(C \setminus C_i) \cup (M \setminus M_i)| = |S_i| \leq n - \sigma_v(G). \]

Hence as $M \cap N(C_i) \subseteq M \setminus M_i$ we have that
\[
|M \cap N(C_i)| \leq |M \setminus M_i| \leq n - \sigma_v(G) - |C \setminus C_i| = |C| + \alpha(G) - \sigma_v(G) - |C \setminus C_i| = |C_i| + \alpha(G) - \sigma_v(G) = |C_i| + \delta(G).
\]

Taking
\[ A_i = C_i \setminus \left( \bigcup_{j=1}^{i-1} C_j \right) \text{ and } B_i = (M \cap N(C_i)) \setminus \bigcup_{j=1}^{i-1} (M \cap N(C_j)), \]
we have that
\[
|C_i \setminus A_i| \leq |M \cap N(C_i) \setminus A_i|,
\]
since if $|C_i \setminus A_i| > |M \cap N(C_i) \setminus A_i|$, then $C \setminus C_i \setminus A_i \cup (M \cap N(C_i \setminus A_i))$ would be a vertex cover of cardinality $|C \setminus C_i \setminus A_i| + |M \cap N(C \setminus A_i)| < |C|$: a contradiction.

To finish the proof, we use the inequalities (5) and (6) to conclude that
\[
|B_i| = |(M \cap N(C_i))\setminus((M \cap N(C_i)) \cap \bigcup_{j=1}^{i-1} (M \cap N(C_j)))| = |(M \cap N(C_i))\setminus(M \cap N(C_i)) \cap N(\bigcup_{j=1}^{i-1} C_j))| \leq |C_i| + \alpha(G) - \sigma_v(G) - |M \cap N(C_i \cap \bigcup_{j=1}^{i-1} C_j))| \leq |C_i| + \alpha(G) - \sigma_v(G) - |C_i \setminus A_i| = |A_i| + \alpha(G) - \sigma_v(G) = |A_i| + \delta(G).
\]

Therefore
\[
\alpha(G) = \sum_{i=1}^{k} (M \cap N(C_i)) = \sum_{i=1}^{k} |B_i| \leq \sum_{i=1}^{k} (|A_i| + \delta(G)) \leq |C| + \tau(G)\delta(G) = \tau(G)[1 + \delta(G)] \quad \square
\]
Remark 4.2 If $\delta(G) > 0$, then we have that $\alpha(G) = \tau(G)[1 + \delta(G)]$ if and only if $G$ is formed by a clique $\mathcal{K}_{\tau(G)}$ with each vertex of this clique being the center of a star $\mathcal{K}_{1,\delta(G)+1}$. Furthermore, if $\delta(G) = 0$ and $\alpha(G) = \tau(G)$, then the graph has a perfect matching.

Figure 1: The graph formed by a clique $\mathcal{K}_{\tau(G)}$ with each vertex of this clique being the center of a star $\mathcal{K}_{1,\delta(G)+1}$.

Let

$$\alpha_{\text{core}}(G) = \bigcap_{|M_i| = \alpha(G)} M_i \quad \text{and} \quad \tau_{\text{core}}(G) = \bigcap_{|C_i| = \tau(G)} C_i,$$

be the intersection of all the maximum stable sets and of all the minimum vertex covers of $G$, respectively.

A graph is $\tau$-critical if $\tau(G \setminus v) < \tau(G)$ for all the vertices $v \in V(G)$, that is, a graph is $\tau$-critical if and only if $\alpha_{\text{core}}(G) = \emptyset$. Similarly, we have that $G$ is a $B$-graph if and only if $\tau_{\text{core}}(G) = \emptyset$. We define $B_{\alpha \land \tau} = V(G) \setminus \{\alpha_{\text{core}}(G) \cup \tau_{\text{core}}(G)\}$.

Proposition 4.3 Let $G$ be a graph, then

$$V(G) = \alpha_{\text{core}}(G) \sqcup \tau_{\text{core}}(G) \sqcup B_{\alpha \land \tau},$$

furthermore

(i) $N(\alpha_{\text{core}}(G)) \subseteq \tau_{\text{core}}(G)$.
(ii) \(G[\alpha_{\text{core}}(G)]\) is a trivial graph,

(iii) \(G[B_{\alpha \cap \tau}]\) is both a \(\tau\)-critical graph as well as a B-graph without isolated vertices.

Proof: Clearly \(\alpha_{\text{core}}(G) \cap \tau_{\text{core}}(G) = \emptyset\). Now, since \(G[V(G) \setminus \tau_{\text{core}}(G)]\) is a B-graph, we have that, \(\alpha_{\text{core}}(G) \subset V(G) \setminus \tau_{\text{core}}(G)\) is the set of isolated vertices of \(G[V(G) \setminus \tau_{\text{core}}(G)]\). Therefore \(N(\alpha_{\text{core}}(G)) \subseteq \tau_{\text{core}}(G)\), proving (i). Hence, we have that \(G[\alpha_{\text{core}}(G)]\) is a graph without edges, giving (ii). Finally, by definition of \(B_{\alpha \cap \tau}\) we obtain (iii).

Example 4.4 To illustrate the previous result consider the following graph:

![Graph Image]

since \(\alpha(G) = 3\), \(\tau(G) = 4\) and \({v_3, v_4, v_5}\), \({v_3, v_4, v_6}\), \({v_3, v_4, v_7}\) are the maximum stable sets of \(G\), we have that

- \(\alpha_{\text{core}}(G) = \{v_3, v_4\}\),
- \(\tau_{\text{core}}(G) = \{v_1, v_2\}\),
- \(B_{\alpha \cap \tau} = \{v_5, v_6, v_7\}\).

Remark 4.5 It is easy to see that if \(v\) is an isolated vertex, then \(v \in \alpha_{\text{core}}(G)\), in a similar way we have that if \(\deg(v) > \tau(G)\), then \(v\) does not belong to any stable set with \(\alpha(G)\) vertices and therefore \(v \in \tau_{\text{core}}(G)\). Note that in general the induced graph \(G[B_{\alpha \cap \tau}]\) is not necessarily connected.

Corollary 4.6 Let \(G\) be a graph, then

\[\alpha(G) - |\alpha_{\text{core}}(G)| \leq \tau(G) - |\tau_{\text{core}}(G)|.\]
Proof: By Proposition 4.3 we have that $G[B_{\alpha \cap \tau}]$ is a $B$-graph. Now, since

$$\alpha(G[B_{\alpha \cap \tau}]) = \alpha(G) - |\alpha_{core}(G)|$$ and $$\tau(G[B_{\alpha \cap \tau}]) = \tau(G) - |\tau_{core}(G)|,$$

and by applying Theorem 4.1 to $G[B_{\alpha \cap \tau}]$ we obtain that

$$\alpha(G) - |\alpha_{core}(G)| \leq \tau(G) - |\tau_{core}(G)|. \quad \square$$

**Remark 4.7** The bound of Corollary 4.6 improves the bound given in [5, Theorem 2.11] for the number of vertices in $\alpha_{core}(G)$. Their result states:

If $G$ is a graph of order $n$ and

$$\alpha(G) > (n + k - \min\{1, |N(\alpha_{core}(G))|\})/2,$$

for some $k \geq 1$, then $|\alpha_{core}(G)| \geq k + 1$. Moreover, if

$$(n + k - \min\{1, |N(\alpha_{core}(G))|\})/2$$

is even, then $|\alpha_{core}(G)| \geq k + 2$.

Notice that if $\alpha(G) \geq n/2 + k'/2$, our bound gives

$$|\alpha_{core}(G)| \geq k' + |\tau_{core}(G)|.$$

**Remark 4.8** After this paper was submitted, the authors learned that Theorem 2.3 was also obtained independently in [2].

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**References**


