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Cooperads as symmetric sequences

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Abstract

We give a brief overview of the basics of cooperad theory using a new definition which lends itself to easy example creation and verification and avoids common pitfalls and complications caused by nonassociativity of the composition operation for cooperads. We also apply our definition to build the parenthesization and cosimplicial structures exhibited by cooperads and give examples.

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1 Introduction

In the current work we discuss cooperads in generic symmetric monoidal categories from the point of view of symmetric sequences. Fix a symmetric monoidal category $(\mathcal{C}, \otimes)$. Let us roughly recall the standard framework.

Operads encode algebra structures. The tautological example is the endomorphism operad of an object $\textsc{end}(A) = \coprod_n \mathrm{Hom}(A^{\otimes n}, A)$. Operads have a natural grading by levels expressing the "arity" of different "operations" (for example, $\textsc{end}(A)(n) = \mathrm{Hom}(A^{\otimes n}, A)$). The symmetric group Σ_n acts on the n-ary operations of an operad (for $\textsc{end}(A)(n)$ this action is by permutation of the $A^{\otimes n}$). A graded object with Σ_n-actions is called a "symmetric sequence." Operads are further equipped with a composition product identifying the result of plugging operations into each other (for example, $\textsc{end}(A) \circ \textsc{end}(A) \to \textsc{end}(A)$). Very roughly, an operad is "a bunch of objects with a rule for plugging them into each other".

Operads encode algebra structures via maps of operads (preserving symmetric group actions and composition structure). So, for example, there is an operad LIE of formal Lie bracket expressions modulo Lie relations, along with a composition rule identifying the result of plugging bracket expressions into each other. A map of operads $\text{LIE} \to \text{END}(A)$ identifies a specific endomorphism of A for each formal Lie bracket expression, in such a way that composition of Lie bracket expressions is compatible with the composition of corresponding endomorphisms. This gives A the structure of a Lie algebra.

Coalgebra structures can also be defined via operads. The coendomorphisms of an object $\text{COEND}(A) = \coprod_n \text{Hom}(A, A^{\otimes n})$ also form an operad: It is graded, with symmetric group action, and has a natural map $\text{COEND}(A) \circ \text{COEND}(A) \to \text{COEND}(A)$ also given by plugging things into each other. Replacing END by COEND changes algebra structures to coalgebra structures. For example a map of operads $\text{LIE} \to \text{COEND}(A)$ identifies a coendomorphism of A for each Lie bracket expression, thus giving A a Lie coalgebra structure.

This is a common point of view (see e.g. [11]), but there is an alternative. For clarity, we will continue with the example of Lie algebras. A Lie algebra structure is maps $\text{LIE}(n) \to \text{Hom}(A^{\otimes n}, A)$ which is equivalent to maps $\text{LIE}(n) \otimes A^{\otimes n} \to A$ (ignore Σ_n-actions for the moment). Dually, a Lie coalgebra structure is maps $\text{LIE}(n) \to \text{Hom}(A, A^{\otimes n})$ which is equivalent to maps $\text{LIE}(n) \otimes A \to A^{\otimes n}$ which is equivalent to $A \to \big(\text{LIE}(n)\big)^* \otimes A^{\otimes n}$. (Dualizing $\text{LIE}(n)$ should not introduce trouble, because it is finite dimensional.) The level-wise dual object $\text{LIE}\check{} = \coprod_n \big(\text{LIE}(n)\big)^*$ has structure dual to that of LIE. This is a cooperad. (The precise definition is the subject of the current paper.)

Experience [9] [10] has shown that it is sometimes more useful to directly work with cooperads and cooperad structures when describing coalgebras rather than continually referring all the way back to operads and operad structures. Also sometimes coalgebras can have a more natural expression as coalgebras over cooperads, rather than coalgebras over operads. Just as operads can be thought of as "a bunch of objects which are plugged into each other", cooperads can be thought of as "a bunch of objects where subobjects are removed or quotiented".

Unfortunately category theory causes a slight hitch when attempting to blindly dualize operad structure to define cooperads. The dual of operad composition is cooperad composition, which is similar except for some colimits being replaced by limits. The problem comes when looking at associativity. In a symmetric monoidal category $\otimes$ is left adjoint (to Hom) so it will commute with colimits. This allows operad composition products to be associative (e.g. $(\text{LIE}\circ\text{LIE})\circ\text{LIE} = \text{LIE}\circ(\text{LIE}\circ$

LIE)). However, this will generally not happen for cooperad composition (e.g. $(\text{LIE}^\vee \bullet \text{LIE}^\vee) \bullet \text{LIE}^\vee \neq \text{LIE}^\vee \bullet (\text{LIE}^\vee \bullet \text{LIE}^\vee)$). This issue crops up for example, in the cooperadic cobar constructions of Ching in his thesis [4] and arXiv note [5].

We work by defining a new composition product – a composition product of tree-functors. The motivating intuition is that *the composition product of two symmetric sequences should not itself be a symmetric sequence* – in particular its group of symmetries is much too large. Maps to and from the tree-functor composition product can be expressed as maps to and from universal extensions, which yields the classical operad and cooperad composition products. Using the tree-functor composition product (rather than its Kan extension) when describing or defining cooperads greatly simplifies bookkeeping; though it turns out that, for operads, it doesn't really make a difference.

We begin by introducing the notation of wreath product categories. These are inspired by the wreath product categories of Berger [2], and at the most basic level are merely Groethendieck constructions. Wreath product categories are defined so that they will be the natural source category for iterated composition products of symmetric sequences. We use this to give a simple definition of cooperads and prove all of the standard structure holds. Then we describe comodules and coalgebras. We finish with simple examples related to work in [9], [10], and [13].

In the sequel [12] we use the structure presented here to build cofree coalgebras, connecting to the constructions of Fox [6] and Smith [11].

We assume that the reader is comfortable with the category theory notions of adjoint functors and Kan extensions, as well as basic simplicial and cosimplicial structures. A familiarity with the classical definitions of operads and their modules/algebras is not required, but would be helpful.

2 Wreath product categories

This section is divided into two parts. In the first subsection, we define wreath product categories using functors to the category of finite sets. Our definition is related to, but more general than, the dual of refined partitions of sets as used in literature by e.g. Arone-Mahowald [1]. The salient difference between wreath categories and refined partitions is that wreath categories incorporate the empty-set (see Remark 2.8). In the second subsection, an equivalent definition is given in terms of labeled level trees – a more familiar category for the discussion of operads.

The neophyte reader (or the reader looking for an immediate connection to classical constructions) may find it useful to read §2.2 **before** §2.1.

2.1 Wreath Products

Write Σ_n for the category of n-element sets and set isomorphisms and $\Sigma_* = \coprod_{n \geq 0} \Sigma_n$ for the category of all finite sets and set isomorphisms ($\Sigma_0 = \emptyset$). Our notation reflects the fact that a functor $\Sigma_n \to \mathcal{C}$ is merely an object of $\mathcal{C}$ with a Σ_n-action.

There is an alternative way to express Σ_n. Write `FinSet` for the category of finite sets and **all** set maps, and write $[\mathtt{n}]$ for the category $1 \xrightarrow{f_1} 2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} n$. Then Σ_* is equivalent to the category of functors $[\mathtt{1}] \to$ `FinSet` and natural isomorphisms. We generalize this to define wreath product categories.

Definition 2.1. The wreath product category $\Sigma_*^{\wr n}$ is the category of contravariant functors $[\mathtt{n}] \to$ `FinSet` and natural isomorphisms.

Remark 2.2. We will write objects of $\Sigma_*^{\wr n}$ as sequences of maps of finite sets, indexed in the following manner.

$$S_1 \xleftarrow{f_1} S_2 \xleftarrow{f_2} \cdots \xleftarrow{f_{n-1}} S_n$$

Since $[\mathtt{n}] \cong [\mathtt{n}]^{\mathrm{op}}$, the use of contravariant functors in Definition 2.1 is purely cosmetic. Using covariant functors would change nothing, except that indices would not line up as perfectly later on.

Note that we are clearly defining the levels of a simplicial category. Before continuing in that direction, however, we will explain our choice of notation via an equivalent, hands-on definition of wreath products with a generic category $\mathcal{A}$.

Definition 2.3. The wreath product category $\Sigma_n \wr \mathcal{A}$ is the category with

- objects $\mathrm{Obj}(\Sigma_n \wr \mathcal{A}) = \big\{ \{A_s\}_{s \in S} \big| \; S \in \mathrm{Obj}(\Sigma_n), \; A_s \in \mathrm{Obj}(\mathcal{A}) \big\}$ given by n-element sets of decorated objects of $\mathcal{A}$;
- and morphisms $\big(\sigma; \; \{\phi_t\}_{t \in T}\big) : \{A_t\}_{t \in T} \longrightarrow \{B_s\}_{s \in S}$ given by a set isomorphism $\sigma : T \to S$ and a set of $\mathcal{A}$-morphisms $\phi_t : A_t \to B_{\sigma(t)}$.

The wreath product category $\Sigma_*\wr\mathcal{A}$ is given by $\Sigma_*\wr\mathcal{A} := \coprod_{n\geq 0}\Sigma_n\wr\mathcal{A}$.

Remark 2.4. $\Sigma_0\wr\mathcal{A}$ is the empty category, since $\{A_s\}_{s\in\emptyset}=\emptyset$. Furthermore $\Sigma_*\wr\Sigma_0\cong\Sigma_*\cong\Sigma_1\wr\Sigma_*$. These equivalences are given by writing objects of $\Sigma_*\wr\Sigma_0$ as $\{\emptyset_s\}_{s\in S}$ and objects of $\Sigma_1\wr\Sigma_*$ as $\{S_\star\}$ and using the facts that $\emptyset$ is initial and a one point set $\star$ is final in `FinSet`. We make further use of these equivalences later. Note that $\Sigma_*\wr\Sigma_1\not\cong\Sigma_*$ because one point sets are not initial in `FinSet`.

The following proposition is easy to check.

Proposition 2.5. *Definitions 2.1 and 2.3 agree:*

- $\Sigma_*^{\wr 2}\cong\Sigma_*\wr\Sigma_*$, *and more generally*
- $\Sigma_*^{\wr n}\cong\Sigma_*\wr\left(\Sigma_*^{\wr n-1}\right)\cong\overbrace{\Sigma_*\wr(\cdots\wr(\Sigma_*\wr\Sigma_*))}^{n}$.

Proof Sketch: The object $\left(S\xleftarrow{f}T\right)$ of $\Sigma_*^{\wr 2}$ corresponds to the object $\left\{f^{-1}(s)_s\right\}_{s\in S}$ of $\Sigma_*\wr\Sigma_*$.

The object $\left\{A_s\right\}_{s\in S}$ corresponds to $\left(S\xleftarrow{\pi}\coprod_{s\in S}A_s\right)$ where π is the map on the coproduct induced by $\pi_s: A_s\to\{s\}\subset S$. □

Using notation from Definition 2.3, the endomorphisms of the wreath product category $\Sigma_n\wr\Sigma_m$ correspond to the automorphisms of an n-element set of m-element sets $S=\{A_1,\ldots,A_n\}$ with $|A_i|=m$. Elements within each A_i can be permuted by Σ_m and the A_i "blocks" are permuted by Σ_n – this is the wreath product group $\Sigma_n\wr\Sigma_m$. Thus, a functor $\Sigma_n\wr\Sigma_m\to\mathcal{C}$ is an object of $\mathcal{C}$ equipped with an action of the wreath product group $\Sigma_n\wr\Sigma_m$. We view $\Sigma_*\wr\Sigma_*$ as a generalization of this basic example – the "blocks" A_i no longer need to be same size, and there can be an arbitrary number of them.

We return to the simplicial structure of the collection of wreath products $\coprod_n\Sigma_*^{\wr n}$. Recall that there are standard "face" functors

$$\partial_i^n : [\mathtt{n}]\to[\mathtt{n-1}]$$

for $1\leq i\leq(n-1)$, given by composing morphisms or forgetting 1 (for reasons to be explained shortly, we do not use the "forget n" face map, ∂_n^n).

$$\partial_1^n(1\xrightarrow{f_1}\cdots\xrightarrow{f_{n-1}}n)=\left(2\to\cdots\to n\right)$$
$$\partial_i^n\left(1\xrightarrow{f_1}\cdots\xrightarrow{f_{n-1}}n\right)=\left(1\to\cdots\to(i-1)\xrightarrow{f_i\circ f_{i-1}}(i+1)\to\cdots\to n\right)$$

Furthermore, (because we do not allow the use of ∂_n^n functors) any chain of $(n-1)$ compositions $\partial_{i_2}^2 \circ \cdots \circ \partial_{i_n}^n : [\mathtt{n}] \to [\mathtt{1}]$ equals the functor $\gamma^n : [\mathtt{n}] \to [\mathtt{1}]$ which forgets all but the top object.

$$\gamma^n\big(1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} n\big) = \big(n\big)$$

We will write ∂_i^n and γ^n also for the induced functors $\partial_i^n : \Sigma_*^{\wr n} \to \Sigma_*^{\wr(n-1)}$, for $1 \leq i \leq (n-1)$, and $\gamma^n : \Sigma_*^{\wr n} \to \Sigma_*$. When n is clear from context we may write merely ∂_i and γ.

Remark 2.6. In the notation of Definition 2.3, the map $\gamma^2 = \partial_1^2 : \Sigma_* \wr \Sigma_* \to \Sigma_*$ is given by $\{S_t\}_{t\in T} \mapsto \coprod_T S_t$. All other ∂_i^n and γ^n are induced by this (see Proposition 2.10).

Before describing the degeneracy maps, we explain the missing ∂_n^n. Recall that Σ_* is equivalent to the full subcategory $\widetilde{\Sigma}_* = \Sigma_1 \wr \Sigma_* \subset \Sigma_* \wr \Sigma_*$ of functors sending 1 to a one-element set. More generally, $\Sigma_*^{\wr n}$ is equivalent to the full subcategory $\widetilde{\Sigma}_*^{\wr n} = \Sigma_1 \wr \Sigma_*^{\wr n} \subset \Sigma_*^{\wr n+1}$ of functors sending 1 to a one-element set. Objects of $\widetilde{\Sigma}_*^{\wr n}$ are sequences of set maps

$$\star \xleftarrow{f_0} S_1 \xleftarrow{f_1} \cdots \xleftarrow{f_{n-1}} S_n$$

Under this correspondence the face functors $\tilde{\partial}_i^n : \widetilde{\Sigma}_*^{\wr n} \to \widetilde{\Sigma}_*^{\wr n-1}$, for $1 \leq i \leq (n-1)$, are all given by composition; however the functor $\tilde{\partial}_n^n$ is not.

$$\tilde{\partial}_i^n\big(\star \xleftarrow{f_0} S_1 \xleftarrow{f_1} \cdots \xleftarrow{f_{n-1}} S_n\big) =$$
$$\big(\star \leftarrow \cdots \leftarrow S_{i-1} \xleftarrow{f_{i-1}\circ f_i} S_{i+1} \leftarrow \cdots \leftarrow S_n\big)$$

(Our indexing convention is for the one point set to be $\star = S_0$ in $\widetilde{\Sigma}_*^{\wr n}$).

Operad and cooperad structure is induced by structure of $\widetilde{\Sigma}_*^{\wr n}$ and $\tilde{\partial}_i^n$. Instead of working with this directly, we use the equivalent categories and functors $\Sigma_*^{\wr n}$ and ∂_i^n; because in practice keeping track of the final, one point set at the bottom of each sequence is unnecessarily tedious.

We continue with the degeneracies of the simplicial structure, which are most conveniently written via the equivalent categories $\widetilde{\Sigma}_*^{\wr n}$. In this notation, the degeneracy functors $\tilde{s}_i^n : \widetilde{\Sigma}_*^{\wr n} \to \widetilde{\Sigma}_*^{\wr n+1}$ for $0 \leq i \leq n$ are the doubling maps.

$$\tilde{s}_i^n\big(\star \xleftarrow{f_0} S_1 \xleftarrow{f_1} \cdots \xleftarrow{f_{n-1}} S_n\big) = \big(\star \leftarrow \cdots \leftarrow S_i \xleftarrow{\mathrm{Id}} S_i \leftarrow \cdots \leftarrow S_n\big)$$

Note that defining the degeneracy s_0^n on the level of $\Sigma_*^{\wr n}$ requires picking a distinguished one point set. A reader averse to making choices should replace all Σ_*, ∂_i, etc. by $\widetilde{\Sigma}_*$, $\tilde{\partial}_i$, etc. from now on.

It is classical that the degeneracies s_{i-1}^n and s_i^n are each sections of the face map ∂_i^{n+1} on the level of $[n]$. Thus face and degeneracy maps combine to give a collection of categories and functors:

$$\cdots \;\substack{\dashleftarrow\\ \rightleftarrows\\ \rightleftarrows\\ \rightleftarrows\\ \rightleftarrows\\ \dashleftarrow}\; \Sigma_* \wr \Sigma_* \wr \Sigma_* \wr \Sigma_* \;\substack{\dashleftarrow\\ \rightleftarrows\\ \rightleftarrows\\ \rightleftarrows\\ \dashleftarrow}\; \Sigma_* \wr \Sigma_* \wr \Sigma_* \;\substack{\dashleftarrow\\ \rightleftarrows\\ \rightleftarrows\\ \dashleftarrow}\; \Sigma_* \wr \Sigma_* \;\substack{\dashleftarrow\\ \rightarrow\\ \dashleftarrow}\; \Sigma_*$$

where the dashed, left-pointing arrows are sections of their neighboring right-pointing arrows and all pairs of neighboring right-pointing arrows are coequalized by an arrow out of their target. Under the correspondence $\Sigma_*^{\wr n} \cong \widetilde{\Sigma}_*^{\wr n} \subset \Sigma_*^{\wr n+1}$, this is very explicitly a simplicial category with the bottom level as well as the first and last face maps removed; equivalently, an augmented simplicial category with two extra degeneracies.

Remark 2.7. We could express **all** of the standard face maps ∂_i^n, $1 \leq i \leq n$, as compositions by writing $\Sigma_*^{\wr n} \cong \overline{\Sigma}_*^{\wr n} = \Sigma_1 \wr \Sigma_*^{\wr n} \wr \Sigma_0 \subset \Sigma_*^{\wr n+2}$, the full subcategory of functors sending $(n+2)$ to the empty-set and 1 to a one-element set. Then ∂_n^n becomes:

$$\overline{\partial}_n^n\big(\star \xleftarrow{f_0} S_1 \xleftarrow{f_1} \cdots \xleftarrow{f_{n-1}} S_n \xleftarrow{f_n} \emptyset\big) =$$

$$\big(\star \leftarrow S_1 \leftarrow \cdots \leftarrow S_{n-1} \xleftarrow{f_{n-1}\circ f_n} \emptyset\big)$$

The $\overline{\Sigma}_*^{\wr n}$ fit together to make an (unaugmented) simplicial category with two extra degeneracies. In the next section, the levels of this will be given an alternate definition and called $\hat{\emptyset}_n$. This structure is useful for constructing algebras and coalgebras instead of operads and cooperads.

Remark 2.8. Another construction which has been useful in the past for describing and working with operads uses the category of sets equipped with iterated refinements of partitions where morphisms are given by set isomorphisms respecting all partition equivalences (see Arone-Mahowald [1] and Ching [4]) . A partition of a set S is equivalent to a **surjective** set map $S \to T$ where T is the partition set. An iterated partition of a set S is equivalent to a functor from $[\mathtt{n}]$ to the category of finite sets and **surjections** (instead of the category of finite sets and **all** set maps). This is sufficient for describing operads and cooperads which are trivial in "0-arity". So partitions cannot be used to describe, for example, an operad of algebras over an algebra. Also missing 0-arity means that partitions cannot work with algebras (or coalgebras) as just a special case of modules (or comodules).

Before continuing with the next subsection, we will combine Definitions 2.1 and 2.3 to get a more general definition of wreath products with generic categories, necessary to discuss associativity.

Definition 2.9. The wreath product category $\Sigma_*^{\wr n} \wr \mathcal{A}$ is the category with

- $\mathrm{Obj}(\Sigma_*^{\wr n} \wr \mathcal{A}) = \Big\{\big(F,\ \{A_s\}_{s\in F(n)}\big) \mid F \in \mathrm{Obj}(\Sigma_*^{\wr n}),\ A_s \in \mathrm{Obj}(\mathcal{A})\Big\}$
- morphisms $\big(\Phi;\ \{\phi_s\}_{s\in F(n)}\big) : \big(F,\ \{A_s\}\big) \to \big(G,\ \{B_t\}\big)$ given by a natural isomorphism $\Phi : F \to G$ and a set of $\mathcal{A}$-morphisms $\phi_s : A_s \to B_{(\Phi n)(s)}$

Objects of $\Sigma_*^{\wr n} \wr \mathcal{A}$ can be written as sequences of set maps

$$S_1 \xleftarrow{f_1} S_2 \xleftarrow{f_2} \cdots \xleftarrow{f_{n-1}} S_{n-1} \xleftarrow{f_n} \{A_s\}_{s\in S_n}.$$

Morphisms are level-wise set isomorphisms accompanied by (at the top level) $\mathcal{A}$-maps $A_s \to B_t$ (where $\phi : S_n \xrightarrow{\cong} T_n$ with $\phi(s) = t$). In the following subsection we will give an alternate way to describe these objects via labeled trees.

Proposition 2.10. *Wreath product is associative:*

$$(\Sigma_* \wr \Sigma_*) \wr \Sigma_* \cong \Sigma_* \wr (\Sigma_* \wr \Sigma_*) \cong \Sigma_*^{\wr 3}.$$

More generally, $\Sigma_*^{\wr n} \wr \Sigma_*^{\wr m} \cong \Sigma_*^{\wr n+m}$. *Furthermore, the face maps* ∂_i^n *are all induced by* $\gamma^2 = \partial_1^2$ *as*

$$\partial_i^n = \mathrm{Id} \wr \gamma^2 \wr \mathrm{Id}\ :\ \Sigma_*^{\wr i-1} \wr \big(\Sigma_* \wr \Sigma_*\big) \wr \Sigma_*^{\wr n-i-1} \longrightarrow \Sigma_*^{\wr i-1} \wr \big(\Sigma_*\big) \wr \Sigma_*^{\wr n-i-1}.$$

For example $\partial_1^3 = \gamma^2 \wr \mathrm{Id}$ and $\partial_2^3 = \mathrm{Id} \wr \gamma^2$.

2.2 Level trees

In this subsection we connect the wreath product constructions of the previous subsection with the classical, visual, method of describing operads via trees.

For our purposes a tree is a (nonempty) non-cyclic, connected, finite graph whose vertices are distinguished as: a "root vertex" of valency 1, a (possibly empty) set of "leaf vertices" of valency 1, and all other vertices called "interior vertices". We require each tree to have a root and at least one interior vertex; however, we do not require that interior vertices have valency > 1 – despite the oxymoron (in particular, we allow the tree with a root, an "interior vertex" but no leaves as in Figure 1). A tree isomorphism is an isomorphism of vertex and edge sets, preserving root and leaf distinctions.

For convenience of notation we will orient all edges of our trees so that they point towards the root vertex; when drawing trees, we will not explicitly indicate this orientation, but rather always position the root at the bottom and the leaves at the top, with the understanding that all edges point downwards. We will denote interior vertices with a darkened dot $\bullet$, but we will not draw the root or leaf vertices – instead we will indicate only the edges connecting to them. Also for convenience, we will draw trees on the plane, however we consider them as non-planar objects. In particular, we will not assert any planar orderings on vertices or edges.

There is a natural height function on the vertices of trees – assigning to each vertex the number of vertices on the path between it and the root (the vertex adjacent to the root has height 0; the root has height -1). A "level n tree" is a tree whose leaves all have height n and whose interior vertices have height $< n$. A "level tree" is a tree which is level n for some n. Note that a level n tree may have branches without leaves which contain no interior vertices of height $(n-1)$, as in Figure 1. In particular, a tree with no leaves may be level n as well as level $(n+1)$, etc.

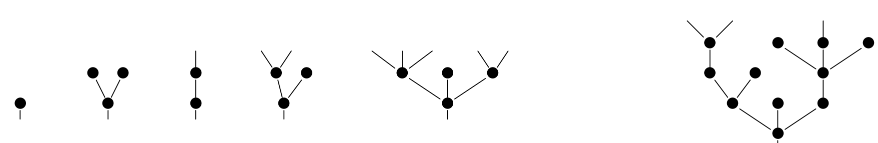


Figure 1: Some examples of level 2 trees and a level 4 tree

If v is the target of the directed edge e then we say e is an "incoming edge" of v and we write $\mathrm{In}(v)$ for the set of incoming edges of v. In our drawings, incoming edges are edges abutting a vertex from above. Each non-root vertex also has one "outgoing edge" (the abutting edge on the path from the vertex to the root), which will be drawn connecting to the vertex from below.

Definition 2.11. A labeled level tree is a level tree equipped with labeling isomorphisms $\{l_v : S_v \xrightarrow{\cong} \mathrm{In}(v)\}_v$ from finite sets to the sets of incoming edges at each vertex. Let Ψ be the category of all labeled level trees with morphisms given by tree isomorphisms. Let Ψ_n be the full subcategory of Ψ consisting of only level n trees.

Since there is always only one incoming edge at the root, and never any incoming edges at leaves, we may equivalently label only the incoming edges at interior vertices.

Definition 2.12. Given a category $\mathcal{A}$ define the wreath product category $\Psi \wr \mathcal{A}$ to be the category of all labeled level trees whose leaves are decorated by elements of $\mathcal{A}$; morphisms are given by tree isomorphisms equipped with $\mathcal{A}$-morphisms between the leaf decorations compatible with the induced isomorphism of leaf sets. Let $\Psi_n \wr \mathcal{A}$ be the full subcategory of this consisting of only level n trees.

Figure 2: Some objects of Ψ_1 and of $\Psi_1 \wr \mathcal{A}$

It is standard to note that the category Σ_* may be identified with the category Ψ_1 of labeled level 1 trees. In this vein, the wreath product category $\Sigma_* \wr \mathcal{A}$ may be identified with $\Psi_1 \wr \mathcal{A}$. More generally, the wreath product category $\Sigma_* \wr \Sigma_*$ is equivalent to the category Ψ_2 of all labeled level 2 trees; and the iterated wreath product category $\Sigma_*^{\wr n}$ is equivalent to Ψ_n the category of all labeled level n trees.

Proposition 2.13. *There are equivalences of categories:*

$$\Psi_1 \cong \Sigma_*, \quad \Psi_1 \wr \mathcal{A} \cong \Sigma_* \wr \mathcal{A}, \quad \Psi_n \cong \Sigma_*^{\wr n}, \quad \textit{and} \quad \Psi_n \wr \mathcal{A} \cong \Sigma_*^{\wr n} \wr \mathcal{A}.$$

Example 2.14. The elements of $\widetilde{\Sigma}_*^{\wr 2}$ corresponding to the Ψ_2 elements in Figure 3 are given by the following chains of maps in `FinSet`:

- $(\star \leftarrow \emptyset \leftarrow \emptyset)$.

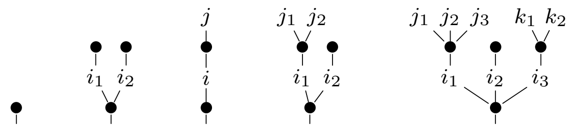


Figure 3: Some objects of Ψ_2

- $\big(\star \leftarrow \{i_2, i_2\} \leftarrow \emptyset\big)$.
- $\big(\star \leftarrow \{i\} \overset{f}{\leftarrow} \{j\}\big)$ where $f(j) = i$.
- $\big(\star \leftarrow \{i_1, i_2\} \overset{f}{\leftarrow} \{j_1, j_2\}\big)$ where $f(j_s) = i_1$.
- $\big(\star \leftarrow \{i_1, i_2, i_3\} \overset{f}{\leftarrow} \{j_1, j_2, j_3, k_1, k_2\}\big)$ where $f(j_s) = i_1$ while $f(k_t) = i_3$.

Under this identification, the functor $\gamma^2 = \partial_1^2 : \Psi_2 \to \Psi_1$ operates by forgetting the height 1 vertices on a level 2 tree. Paths from the height 0 interior vertex to leaves (on level 2) are replaced by edges; the labeling of each such edge is given by the path labeling of the path which it replaces, as in Figure 4.

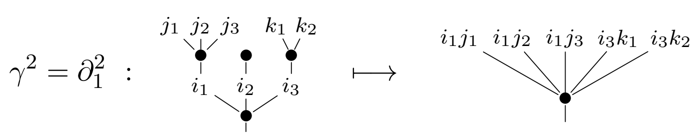


Figure 4: An example of $\gamma^2 : \Psi_2 \to \Psi_1$

Similarly, the functors $\gamma^n : \Psi_n \to \Psi_1$ operate by forgetting all interior vertices except for those of height 0; replacing paths by edges carrying the paths' labels. The face functors $\partial_i^n : \Psi_n \to \Psi_{n-1}$ for $1 \le i \le n-1$ are given by forgetting only the vertices of level i of a level n tree. (The disallowed face functor ∂_n^n would forget the leaves.) The degeneracy functors $s_i^n : \Psi_n \to \Psi_{n+1}$ for $0 \le i \le n$ are given by "doubling" – replace each vertex v at level i by two vertices connected by a directed edge e_v, attached to the tree such that all incoming edges connect to source vertex of e_v and the outgoing edge connects to the target vertex (for the labeling, allow each edge to label itself $l_{t(e_v)} : \{e_v\} \to \{e_v\}$). Note that the degeneracy s_n^n doubles the leaf vertices – the leaves of the resulting tree are the sources of the edges e_v.

Remark 2.15. We very purposefully do not use the notation Υ for our category of level trees, since that notation is already commonly used

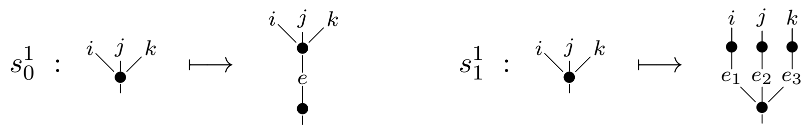


Figure 5: An example of $s_0^1, s_1^1 : \Psi_1 \to \Psi_2$

to denote the category consisting of all trees. The category Ψ differs from this both on the level of objects (only level trees) and on the level of morphisms (only isomorphisms of trees – in particular, no "edge contraction" maps).

Remark 2.16. Note that Ψ is not isomorphic to the category $\Psi_* = \coprod_n \Psi_n$. Write $\hat{\emptyset}_n$ for the full subcategory of Ψ_n consisting of trees with no leaves. Then $\hat{\emptyset}_n$ is a full subcategory of $\hat{\emptyset}_{n+1}$. In terms of the Ψ_n, the category Ψ itself is given by

$$\Psi \cong \Psi_1 \coprod_{\hat{\emptyset}_1} \Psi_2 \coprod_{\hat{\emptyset}_2} \Psi_3 \coprod_{\hat{\emptyset}_3} \Psi_4 \cdots$$

In the notation of the previous subsection, an element of $\hat{\emptyset}_n$ is equivalent to a contravariant functor $[\mathtt{n}] \to \mathtt{FinSet}$ sending n to the empty-set as in Remark 2.7.

3 Symmetric sequences, composition products, and cooperads

3.1 Symmetric Sequences

Let $(\mathcal{C}, \otimes, 1_\otimes)$ be a symmetric monoidal category with monoidal unit $1_\otimes$. In order to have all desired Kan extensions exist, we will further require that $\mathcal{C}$ is cocomplete. Write $\star_\mathcal{C}$ for the final object of $\mathcal{C}$. [In order to dualize to operads, we would require $\mathcal{C}$ be complete with initial object $\emptyset_\mathcal{C}$.]

Definition 3.1. A symmetric sequence is a functor $A : \Sigma_* \to \mathcal{C}$.

Recall that a functor $\Sigma_* \to \mathcal{C}$ is equivalent to a sequence of objects $\{A(n)\}_{n \geq 0}$ of $\mathcal{C}$ along with a symmetric group action on each $A(n)$. We will make use of this viewpoint when convenient without further comment. If A is a symmetric sequence, then we will refer to $A(n)$ as the "n-ary part of A" since for operads it will encode n-ary algebra operations. (The "0-ary operations" require no input. For example, in the category of algebras over a field, elements of the base field are all 0-ary operations.)

3.2 Composition of Symmetric Sequences

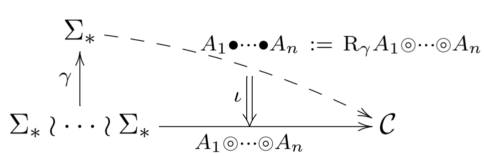
We define a "product" operation on symmetric sequences. It is important to note that our product will not itself be a symmetric sequence. Instead it is a larger diagram, reflecting a larger group of symmetries. The traditional composition product of operads as well as our cooperad composition product are Kan extensions of this symmetric sequence product.

Definition 3.2. Given $A_1, \ldots, A_n : \Sigma_* \to \mathcal{C}$ define $(A_1 \circledcirc \cdots \circledcirc A_n) : \widetilde{\Sigma}_*^{\wr n} \to \mathcal{C}$ by

$$\left(\star \xleftarrow{f_0} S_1 \xleftarrow{f_1} \cdots \xleftarrow{f_{n-1}} S_n\right) \longmapsto \bigotimes_{0 \le i \le n-1} \left(\bigotimes_{s \in S_i} A_{i+1}\left(f_i^{-1}(s)\right) \right)$$

with the convention that $\star = S_0$.

Define $A_1 \bullet \cdots \bullet A_n$ to be the right Kan extension of $A_1 \circledcirc \cdots \circledcirc A_n$ over the map $\gamma : \Sigma_*^{\wr n} \longrightarrow \Sigma_*$.



Write $\iota : (A_1 \bullet \cdots \bullet A_n)\,\gamma \to A_1 \circledcirc \cdots \circledcirc A_n$ for the universal natural transformation.

[Dually, to construct operads , we would define $A_1 \circ \cdots \circ A_n$ to be the <u>left</u> Kan extension over γ.]

Remark 3.3 (For young readers)**.** The symmetric sequence $A \bullet B$ is completely determined by the property that every natural transformation $C\gamma \xrightarrow{\zeta} A \circledcirc B$ factors uniquely as $C\gamma \xrightarrow{\xi} (A \bullet B)\gamma \xrightarrow{\iota} A \circledcirc B$. Dually, $A \circ B$ is determined by the the unique factorization of every transformation $A \circledcirc B \xrightarrow{\rho} D\gamma$ as $A \circledcirc B \xrightarrow{\pi} (A \circ B)\gamma \xrightarrow{\beta} D\gamma$.

Using the notation of Definition 2.9, we can generalize the above definition slightly in order to discuss associativity.

Definition 3.4. Given $A : \Sigma_*^{\wr n} \to \mathcal{C}$ and $B : \mathcal{A} \to \mathcal{C}$, define $(A \circledcirc B) : \Sigma_*^{\wr n} \wr \mathcal{A} \longrightarrow \mathcal{C}$ by

$$\big(A \circledcirc B\big)\Big(F,\ \{A_s\}_{s\in F(n)}\Big) = A(F) \otimes \left(\bigotimes_{s\in F(n)} B(A_s) \right).$$

Remark 3.5 (Completed tensor product)**.** The discussion of coalgebras via cooperads in the introduction glossed over an important subtlety. The dual of the endomorphism operad is

$$\coprod_n \mathrm{Hom}(A^*,\ (A^{\otimes n})^*) = \coprod_n \mathrm{Hom}(A^*,\ (A^*)^{\hat{\otimes} n})$$

where $\hat{\otimes}$ is the "completed" tensor product (if $\mathcal{C}^*$ is a category where this exists). The product $\hat{\otimes}$ is a **right** adjoint (rather than left adjoint) to Hom. In cases where it exists, $\hat{\otimes}$ is usually something like "formal, infinite linear combinations of elements $a \otimes b$". For coalgebras to satisfy a maximal number of duality properties with algebras, the completed tensor product (if it exists) should make an appearance once we begin discussing coalgebras; but it is not used in the construction of cooperads. Note that in categories satisfying good finiteness conditions (for example finitely generated projective bimodules over a commutative ring), $A^* \otimes B^* = (A \otimes B)^* = A^* \hat{\otimes} B^*$.

Short calculations yield the following propositions.

Proposition 3.6. *The operation $\circledcirc$ is associative:*

$$(A_1 \circledcirc A_2) \circledcirc A_3 \ \cong\ A_1 \circledcirc A_2 \circledcirc A_3 \ \cong\ A_1 \circledcirc (A_2 \circledcirc A_3).$$

Proposition 3.7. *Given A, B symmetric sequences, $A \bullet B$ is given by*

$$(A \bullet B)(n) \ = \ \prod_{k\geq 0} \left(\prod_{\sum r_i = n} A(k) \otimes B(r_1) \otimes \cdots \otimes B(r_k) \right)^{\Sigma_k}.$$

Note that $\bullet$ is probably not associative. This will be discussed in greater detail in the next section (see Proposition 4.1). The operation $\circledcirc$ is clearly functorial. If $F : A_1 \to A_2$ and $G : B_1 \to B_2$ are natural transformations of functors $A_1, A_2 : \Sigma_*^{\wr n} \to \mathcal{C}$ and $B_1, B_2 : \Sigma_*^{\wr m} \to \mathcal{C}$, then we write $(F \circledcirc G) : (A_1 \circledcirc B_1) \to (A_2 \circledcirc B_2)$ for the induced natural transformation of functors $\Sigma_*^{\wr n} \wr \Sigma_*^{\wr m} \to \mathcal{C}$.

In the following subsections, we define cooperad structure and, in parallel, build the cosimplicial structure induced on $\coprod_n A^{\circledcirc n}$ by the simplicial structure of wreath product categories $\coprod_n \Sigma_*^{\wr n}$.

3.3 Cocomposition and Coface Maps

Definition 3.8. A symmetric sequence with cocomposition is $(A,\ \tilde{\Delta})$ where $\tilde{\Delta}$ is a cocomposition natural transformation $\tilde{\Delta} : A\,\gamma^2 \longrightarrow A \circledcirc A$ of functors $\Sigma_* \wr \Sigma_* \to \mathcal{C}$ compatible with the face maps $\partial_1^3 = (\gamma^2 \wr \mathrm{Id})$ and $\partial_2^3 = (\mathrm{Id} \wr \gamma^2)$.

Write Δ for the associated universal natural transformation of symmetric sequences $\Delta : A \longrightarrow A \bullet A$.

In other words, the following diagram of functors $\Sigma_* \wr \Sigma_* \wr \Sigma_* \to \mathcal{C}$ should commute.

$$\begin{array}{ccccc} & \overset{\tilde{\Delta}}{\nearrow} & (A \circledcirc A)(\gamma^2 \wr \mathrm{Id}) & \overset{\tilde{\Delta}\wr\mathrm{Id}}{\searrow} & \\ A\gamma^3 & & & & A \circledcirc A \circledcirc A \\ & \underset{\tilde{\Delta}}{\searrow} & (A \circledcirc A)(\mathrm{Id} \wr \gamma^2) & \underset{\mathrm{Id}\wr\tilde{\Delta}}{\nearrow} & \end{array} \tag{1}$$

The upper path uses the factorization $\gamma^3 = \partial_1^2 \circ \partial_1^3 = \gamma^2 \circ (\gamma^2 \wr \mathrm{Id})$ and the lower path uses the factorization $\gamma^3 = \partial_1^2 \circ \partial_2^3 = \gamma^2 \circ (\mathrm{Id} \wr \gamma^2)$.

Applying Proposition 2.10, we may generalize $\tilde{\Delta}$ to the following maps.

Definition 3.9. For a given a symmetric sequence with cocomposition $(A,\ \tilde{\Delta})$ define associated natural transformations $\tilde{\Delta}_i^n : A^{\circledcirc(n-1)}\partial_i^n \to A^{\circledcirc n}$, for $1 \le i \le (n-1)$, which apply $\tilde{\Delta}$ at position i. (Thus $\tilde{\Delta} = \tilde{\Delta}_1^2$.)

These natural transformations induce coface maps on $\coprod_n A^{\circledcirc n}$ in the following manner. Since $\gamma^{n-1}\,\partial_i^n = \gamma^n$ and ∂_i^n is epi, transformations $B\,\gamma^n \to A^{\circledcirc(n-1)}\,\partial_i^n$ are equivalent to transformations $B\gamma^{n-1} \to A^{\circledcirc(n-1)}$ (where $B : \Sigma_* \to \mathcal{C}$ is some symmetric sequence). Therefore there is an equality of right Kan extensions $\mathrm{R}_{\gamma^n}\big(A^{\circledcirc(n-1)}\,\partial_i^n\big) = \mathrm{R}_{\gamma^{n-1}}\big(A^{\circledcirc(n-1)}\big) = A^{\bullet(n-1)}$. (We will make extensive use of this equality in later sections without further comment.)

Define $\Delta_i^n : A^{\bullet(n-1)} \to A^{\bullet n}$ to be the following map.

(2)
$$\begin{array}{ccc} A^{\bullet(n-1)} & & A^{\bullet n} \\ \| & & \| \\ \mathrm{R}_{\gamma^n}\big(A^{\circledcirc(n-1)}\,\partial_i^n\big) & \xrightarrow{\mathrm{R}_{\gamma^n}(\tilde{\Delta}_i^n)} & \mathrm{R}_{\gamma^n}\big(A^{\circledcirc n}\big) \end{array}$$

Under right Kan extension, Diagram (1) translates to the following diagram of symmetric sequences.

(3)
$$\begin{array}{ccccc} & \overset{\Delta}{\nearrow} & A \bullet A & \overset{\Delta_1^3}{\searrow} & \\ A & & & & A \bullet A \bullet A \\ & \underset{\Delta}{\searrow} & A \bullet A & \underset{\Delta_2^3}{\nearrow} & \end{array}$$

Combined with Proposition 2.10, this generalizes to the following.

Proposition 3.10. *Let $(A, \tilde{\Delta})$ be a symmetric sequence with cocomposition. Then the transformation $\Delta_i^n : A^{\bullet(n-1)} \to A^{\bullet n}$ equalizes the two transformations*
$$\Delta_i^{n+1},\, \Delta_{i+1}^{n+1} : A^{\bullet n} \rightrightarrows A^{\bullet(n+1)}.$$
More generally, $\Delta_j^{n+1}\,\Delta_i^n = \Delta_i^{n+1}\,\Delta_{j-1}^n$ for $j > i$.

Corollary 3.11. *Let $(A, \tilde{\Delta})$ be a symmetric sequence with cocomposition. There are canonical, unique maps $\Delta^{[n]} : A \to A^{\bullet n}$. (Given by taking any chain of compositions $\Delta_{i_n}^n \cdots \Delta_{i_1}^1$.)*

3.4 Counit and Codegeneracies

Write $\mathbb{1}$ for the functor $\mathbb{1} : \Sigma_* \to \mathcal{C}$ given by
$$\mathbb{1}(T) = \begin{cases} 1_\otimes & \text{if } |T| = 1, \\ \star_{\mathcal{C}} & \text{otherwise.} \end{cases}$$
We will call $\mathbb{1}$ the "counit" symmetric sequence. [The dual definition of the "unit" symmetric sequence would use $\emptyset_{\mathcal{C}}$.]

Definition 3.12. A counital symmetric sequence is $(A, \tilde{\epsilon})$ where A is a symmetric sequence and $\tilde{\epsilon}$ is a natural transformation to the counit $\tilde{\epsilon} : A \to \mathbb{1}$.

Note that being counital is equivalent to the existence of a map $A(1) \to 1_\otimes$. We will not require the map $A(1) \to 1_\otimes$ to be equipped with a section. In the next subsection, we will use the following basic equality whose proof can be read off of Figure 5.

Lemma 3.13. *The following functors* $\Sigma_* \to \mathcal{C}$ *are equal.*

$$\big(\mathbb{1} \circledcirc A\big) s_0^1 \;=\; A \;=\; \big(A \circledcirc \mathbb{1}\big) s_1^1.$$

More generally, the following functors $\Sigma_*^{\wr n} \to \mathcal{C}$ *are equal.*

$$\Big(\big(A^{\circledcirc i}\big) \circledcirc \mathbb{1} \circledcirc \big(A^{\circledcirc (n-i)}\big)\Big)\, s_i^n = A^{\circledcirc n}$$

In the footsteps of Lemma 3.13 we define the following generalization.

Definition 3.14. Given a counital symmetric sequence $(A,\, \tilde{\epsilon})$ define associated natural transformations $\tilde{\epsilon}_i^n : A^{\circledcirc(n+1)} s_i^n \to A^{\circledcirc n}$, for $0 \leq i \leq n$, to be the following compositions.

$$\begin{array}{ccc} A^{\circledcirc(n+1)} s_i^n & & A^{\circledcirc n} \\ \| & & \| \\ \Big(\big(A^{\circledcirc i}\big) \circledcirc A \circledcirc \big(A^{\circledcirc (n-i)}\big)\Big)\, s_i^n & \xrightarrow{(\mathrm{Id} \circledcirc \tilde{\epsilon} \circledcirc \mathrm{Id})\, s_i^n} & \Big(\big(A^{\circledcirc i}\big) \circledcirc \mathbb{1} \circledcirc \big(A^{\circledcirc (n-i)}\big)\Big)\, s_i^n \end{array}$$

Define $\tilde{\epsilon}_0^0 = \tilde{\epsilon} : A \to \mathbb{1}$.

These natural transformations induce codegeneracies in the following manner. Since $\gamma^{n+1}\, s_i^n = \gamma^n$, the universal transformation

$$A^{\bullet(n+1)}\, \gamma^{n+1} \to A^{\circledcirc(n+1)}$$

induces a transformation $A^{\bullet(n+1)} \to \mathrm{R}_{\gamma^n}\big(A^{\circledcirc(n+1)}\, s_i^n\big)$. Define $\epsilon_i^n : A^{\bullet(n+1)} \to A^{\bullet n}$ to be the following composition.

$$\begin{array}{ccccc} & & & & A^{\bullet n} \\ & & & & \| \\ A^{\bullet(n+1)} & \longrightarrow & \mathrm{R}_{\gamma^n}\big(A^{\circledcirc(n+1)}\, s_i^n\big) & \xrightarrow{\mathrm{R}_{\gamma^n}(\tilde{\epsilon}_i^n)} & \mathrm{R}_{\gamma^n}\big(A^{\circledcirc n}\big) \end{array} \tag{4}$$

Similar to Proposition 3.10, the corresponding properties of s_i^n imply the following.

Proposition 3.15. *Let* $(A, \tilde{\epsilon})$ *be a counital symmetric sequence. Then the transformation* $\epsilon_i^{n-1} : A^{\bullet n} \to A^{\bullet(n-1)}$ *coequalizes the two transformations* $\epsilon_i^n,\, \epsilon_{i+1}^n : A^{\bullet(n+1)} \rightrightarrows A^{\bullet n}$. *More generally* $\epsilon_i^{n-1}\, \epsilon_j^n = \epsilon_{j-1}^{n-1}\, \epsilon_i^n$ *for* $j > i$.

3.5 Cooperads and Cosimplicial Structure

Definition 3.16. A cocomposition operation on a counital symmetric sequence respects the counit if the following diagram of natural transformations $\Sigma_* \to \mathcal{C}$ commutes.

$$\begin{array}{ccccccccc} & \overset{=}{\nearrow} & A\gamma^2 s_0^1 & \xrightarrow{\tilde{\Delta} s_0^1} & (A \circledcirc A) s_0^1 & \xrightarrow{(\tilde{\epsilon}\circledcirc \mathrm{Id}) s_0^1} & (\mathbb{1} \circledcirc A) s_0^1 & \overset{=}{\searrow} & \\ A & & & & \xrightarrow{\quad\mathrm{Id}\quad} & & & & A \\ & \underset{=}{\searrow} & A\gamma^2 s_1^1 & \xrightarrow[\tilde{\Delta} s_1^1]{} & (A \circledcirc A) s_1^1 & \xrightarrow[(\mathrm{Id}\circledcirc\tilde{\epsilon}) s_1^1]{} & (A \circledcirc \mathbb{1}) s_1^1 & \underset{=}{\nearrow} & \end{array} \tag{5}$$

A counital cooperad is a counital symmetric sequence with cocomposition which respects the counit.

By applying Proposition 2.10 and using the simplicial structure of wreath product categories, the requirement in Definition 3.16 implies a more general statement.

Proposition 3.17. *If $(\mathcal{O}, \tilde{\Delta}, \tilde{\epsilon})$ is a cooperad, then the following composition is equal to the identity $\mathrm{Id}_{\mathcal{O}^{\circledcirc n}}$, for $j = (i-1), i$.*

$$\mathcal{O}^{\circledcirc n} = \mathcal{O}^{\circledcirc n}\, \partial_i^{n+1} s_j^n \xrightarrow{\tilde{\Delta}_i^{n+1}\, s_j^n} \mathcal{O}^{\circledcirc (n+1)}\, s_j^n \xrightarrow{\tilde{\epsilon}_j^n} \mathcal{O}^{\circledcirc n}$$

Furthermore, the following compositions are equal if $j < i-1$.

$$\begin{array}{ccccc} \mathcal{O}^{\circledcirc n}\, (\partial_i^{n+1}\, s_j^n) & \xrightarrow{\tilde{\Delta}_i^{n+1}\, s_j^n} & \mathcal{O}^{\circledcirc (n+1)}\, s_j^n & \xrightarrow{\tilde{\epsilon}_j^n} & \mathcal{O}^{\circledcirc n} \\ \| & & & & \\ \mathcal{O}^{\circledcirc n}\, (s_j^{n-1} \partial_{i+1}^n) & \xrightarrow[\tilde{\epsilon}_j^{n-1}\, \partial_{i+1}^n]{} & \mathcal{O}^{\circledcirc (n-1)} \partial_{i+1}^n & \xrightarrow[\tilde{\Delta}_{i+1}^n]{} & \mathcal{O}^{\circledcirc n} \end{array}$$

as well as the similar statement for $j > i$.

We have now almost completed the proof of the following.

Theorem 3.18. *If $(\mathcal{O}, \tilde{\Delta}, \tilde{\epsilon})$ is a cooperad, then the collection $\{\mathcal{O}^{\bullet n}\}_n$ along with coface maps Δ_i^n and codegeneracy maps ϵ_i^n defines a coaugmented cosimplicial symmetric sequence with two extra codegeneracies.*

$$\mathcal{O} \rightleftarrows \mathcal{O} \bullet \mathcal{O} \rightleftarrows \mathcal{O}^{\bullet 3} \rightleftarrows \mathcal{O}^{\bullet 4} \rightleftarrows \cdots \tag{6}$$

Proof. In Propositions 3.10 and 3.15, we have already shown the cosimplicial identities $\Delta_j^{n+1}\,\Delta_i^n = \Delta_i^{n+1}\,\Delta_{j-1}^n$ and $\epsilon_i^{n-1}\,\epsilon_j^n = \epsilon_{j-1}^{n-1}\,\epsilon_i^n$.

It remains only to consider the compositions $\Delta_i^{n+1}\,\epsilon_j^n$. These come from the right Kan extension over γ^n of the statements of Proposition 3.17. Note that the right Kan extension

$$\mathrm{R}_{\gamma^n}\Big(\mathcal{O}^{\circledcirc n}\,\partial_i^{n+1} s_j^n \xrightarrow{\tilde{\Delta}_i^{n+1}\,s_j^n} \mathcal{O}^{\circledcirc(n+1)}\,s_j^n\Big)$$

is equal to the composition

$$\mathrm{R}_{\gamma^{n+1}}\big(\mathcal{O}^{\circledcirc n}\,\partial_i^{n+1}\big) \xrightarrow{\Delta_i^{n+1}} \mathrm{R}_{\gamma^{n+1}}\big(\mathcal{O}^{\circledcirc(n+1)}\big) \longrightarrow \mathrm{R}_{\gamma^n}\big(\mathcal{O}^{\circledcirc(n+1)}\,s_j^n\big).$$

□

Corollary 3.19. *There are unique transformations $\Delta^{[n]} : \mathcal{O} \to \mathcal{O}^{\bullet n}$. These are equal to any combination of parenthesization maps and cocomposition maps from their source to their target.*

4 Cooperads via • versus ⊚

We will now connect the cooperad structures defined in the previous sections with the classical methods which would attempt to use only the induced product • on symmetric sequences. When using •, the lack of associativity introduces extra "parenthesization" maps, which must be dealt with carefully.

4.1 Parenthesization Maps.

From now on, let A, B, C be generic symmetric sequences and $(\mathcal{O},\, \tilde{\Delta},\, \tilde{\epsilon})$ be a generic counital cooperad.

Proposition 4.1. *There are canonical "parenthesization" natural transformations:*

$$(A \bullet B) \bullet C \longrightarrow A \bullet B \bullet C \longleftarrow A \bullet (B \bullet C)$$

More generally there are parenthesization maps to $A_1 \bullet \cdots \bullet A_n$ from any parenthesization of this expression.

Proof. We show the existence of the map $(A \bullet B) \bullet C \to A \bullet B \bullet C$. The other maps are similar.

The universal natural transformation $(A \bullet B)\,\gamma^2 \longrightarrow (A \circledcirc B)$ induces a natural transformation of functors $(\Sigma_* \wr \Sigma_*) \wr \Sigma_* \longrightarrow \mathcal{C}$:

$$\big((A \bullet B) \circledcirc C\big)\partial_1^3 \longrightarrow (A \circledcirc B) \circledcirc C = A \circledcirc B \circledcirc C.$$

The desired map is induced by taking the right Kan extension R_{γ^3} of the diagram above.

$$\begin{array}{ccc} (A \bullet B) \bullet C & & A \bullet B \bullet C \\ \| & & \| \\ \mathrm{R}_{\gamma^3}\Big(\big((A \bullet B) \circledcirc C\big)\,\partial_1^3\Big) & \longrightarrow & \mathrm{R}_{\gamma^3}(A \circledcirc B \circledcirc C) \end{array}$$ □

Remark 4.2 (On the associativity of $\bullet$)**.** Without making further assumptions, it is **not** true that $(A \bullet B) \bullet C \cong A \bullet B \bullet C \cong A \bullet (B \bullet C)$. This would follow from the existence of natural equivalences $\big(\mathrm{R}_{\gamma^2}(A \circledcirc B)\big) \circledcirc C \cong \mathrm{R}_{\partial_1^3}(A \circledcirc B \circledcirc C)$ as well as the corresponding equivalence using ∂_2^3. However, this will generally only occur in the unlikely event of the symmetric monoidal product commuting with categorical products.

The situation contrasts starkly with that of the operad composition product, defined dual to $\bullet$ using left rather than right Kan extensions. If $\mathcal{C}$ is a closed monoidal category, then $\otimes$ is a left adjoint, so it will in particular commute with categorical coproducts and left Kan extensions. In this case the parenthesization maps for the operad composition product are isomorphisms and the operad composition product is associative.

In practice, authors have generally dealt with this in the past by either not using cooperads at all, or by (implicitly or explicitly) restricting their categories so that $(\cdots)^{\Sigma_n} = (\cdots)_{\Sigma_n}$ and $\prod = \coprod$. In this (very special case) $A \bullet B = A \circ B$ and there is no problem. Alternately, heavy restrictions can be placed on $\mathcal{C}$ and/or on the category of symmetric sequences to force $\otimes$ to commute with $\prod$. For example, in the category of finitely generated, injective bimodules over a commutative ring, $\otimes = \hat{\otimes}$ which is a right adjoint.

Proposition 4.3. *Parenthesization maps are associative.*

For example the following diagrams commute.

$$\begin{array}{ccccc} & \nearrow & (A \bullet B \bullet C) \bullet D & \searrow & \\ \big((A \bullet B) \bullet C\big) \bullet D & & & & A \bullet B \bullet C \bullet D \\ & \searrow & (A \bullet B) \bullet C \bullet D & \nearrow & \end{array} \tag{7}$$

$$
(8)\qquad
\begin{array}{ccccc}
 & \nearrow & (A\bullet B)\bullet C\bullet D & \searrow & \\
(A\bullet B)\bullet(C\bullet D) & & & & A\bullet B\bullet C\bullet D \\
 & \searrow & A\bullet B\bullet(C\bullet D) & \nearrow &
\end{array}
$$

Proof of 4.3. It is enough to consider Diagrams (7) and (8). Commutativity is shown by writing the diagrams as right Kan extensions. The diagrams above are R_{γ^4} of the following diagrams of functors $\Sigma_*^{\wr 4} \to \mathcal{C}$.
(7')

$$
\begin{array}{ccccc}
 & \nearrow & \big((A\bullet B\bullet C)\circledcirc D\big)\,(\gamma^3\wr \mathrm{Id}) & \searrow & \\
\Big(\big((A\bullet B)\bullet C\big)\circledcirc D\Big)\,(\gamma^3\wr \mathrm{Id}) & & & & A\circledcirc B\circledcirc C\circledcirc D \\
 & \searrow & \big((A\bullet B)\circledcirc C\circledcirc D\big)\,\partial^4_1 & \nearrow &
\end{array}
$$

(8')

$$
\begin{array}{ccccc}
 & \nearrow & \big((A\bullet B)\circledcirc C\circledcirc D\big)\,\partial^4_1 & \searrow & \\
\big((A\bullet B)\circledcirc (C\bullet D)\big)\,(\gamma^2\wr \mathrm{Id}\wr\gamma^2) & & & & A\circledcirc B\circledcirc C\circledcirc D \\
 & \searrow & \big(A\circledcirc B\circledcirc (C\bullet D)\big)\,\partial^4_3 & \nearrow &
\end{array}
$$

Diagram (7') is just $-\circledcirc D$ applied to the following universal diagram (in which the upper-left map is R_{γ^3} of the lower-right).

$$
(7'')\qquad
\begin{array}{ccccc}
 & \nearrow & (A\bullet B\bullet C)\,\gamma^3 & \searrow & \\
\big((A\bullet B)\bullet C\big)\,\gamma^3 & & & & A\circledcirc B\circledcirc C \\
 & \searrow & \big((A\bullet B)\circledcirc C\big)\,\partial^3_1 & \nearrow &
\end{array}
$$

Diagram (8') commutes because the upper and lower composition are both equal to

$$\big((A\bullet B)\circledcirc (C\bullet D)\big)\,(\gamma^2\wr \mathrm{Id}\wr\gamma^2) \xrightarrow{\iota_1\circledcirc\iota_2} (A\circledcirc B)\circledcirc (C\circledcirc D)$$

Where $\iota_1 : (A\bullet B)\,\gamma^2 \to A\circledcirc B$ and $\iota_2 : (C\bullet D)\,\gamma^2 \to C\circledcirc D$ are the universal natural transformations from their respective Kan extensions.

□

4.2 Cooperad Structures

We relate parenthesization maps with cooperad structure. By the functoriality of $\circledcirc$, there are natural transformations

$$\mathrm{Id}\circledcirc\Delta : A\circledcirc\mathcal{O}\to A\circledcirc(\mathcal{O}\bullet\mathcal{O}) \quad\text{and}\quad \Delta\circledcirc\mathrm{Id} : \mathcal{O}\circledcirc A\to(\mathcal{O}\bullet\mathcal{O})\circledcirc A,$$

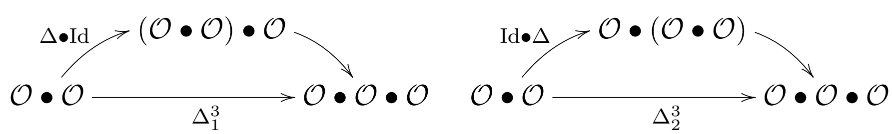
where A is any symmetric sequence. Define the maps $\mathrm{Id} \bullet \Delta$ and $\Delta \bullet \mathrm{Id}$ to be the natural transformations induced on right Kan extensions via functoriality of Kan extension. For example

$$\Delta \bullet \mathrm{Id} = \mathrm{R}_{\gamma^2}\big(\Delta \circledcirc \mathrm{Id}\big) : \mathcal{O} \bullet A \longrightarrow (\mathcal{O} \bullet \mathcal{O}) \bullet A.$$

By alternately letting A be a parenthesization of $\mathcal{O}^{\bullet k}$ and using functoriality of $\bullet$ this defines maps originating in any parenthesization of $\mathcal{O}^{\bullet n}$. For example

$$\big((\mathrm{Id} \bullet \mathrm{Id}) \bullet \Delta\big) \bullet \mathrm{Id} : \big((\mathcal{O} \bullet \mathcal{O}) \bullet \mathcal{O}\big) \bullet \mathcal{O} \longrightarrow \big((\mathcal{O} \bullet \mathcal{O}) \bullet (\mathcal{O} \bullet \mathcal{O})\big) \bullet \mathcal{O}.$$

Theorem 4.4. *The following diagrams commute (unlabeled maps are parenthesization):*

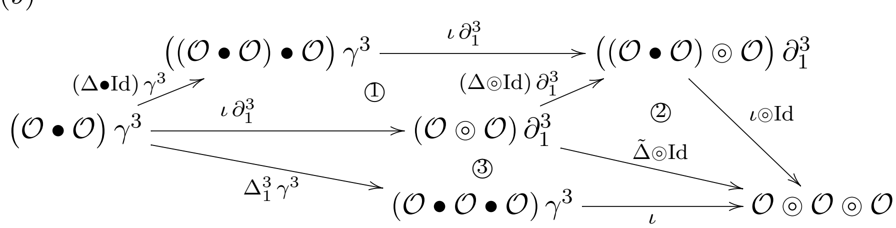


More generally, parenthesization maps convert $\mathrm{Id} \bullet \Delta \bullet \mathrm{Id}$ *(and its parenthesizations) to* Δ^4_2*, etc.*

Proof. We show the first diagram commutes. The second diagram and more general statement are proven the same.

Consider the diagram below, where maps marked ι are all universal transformations of right Kan extensions $(\mathrm{R}_F X)\, F \xrightarrow{\iota} X$.

(9)



Parallelograms ① and ③ commute by functoriality of right Kan extension. The left side of parallelogram ① is R_{γ^2} of the right side, and the left side of parallelogram ③ is R_{γ^3} of the right side. Triangle ② commutes by functoriality of $\circledcirc$ (recall that $\iota\Delta = \tilde{\Delta}$).

Applying R_{γ^3} along the outside of Diagram (9) yields the following (where the map labeled $*$ is the parenthesization map):

$$
\begin{array}{ccccc}
 & (\mathcal{O} \bullet \mathcal{O}) \bullet \mathcal{O} & \xrightarrow{=} & (\mathcal{O} \bullet \mathcal{O}) \bullet \mathcal{O} & \\
{\scriptstyle \Delta \bullet \mathrm{Id}} \nearrow & & & & \searrow {\scriptstyle *} \\
\mathcal{O} \bullet \mathcal{O} & \xrightarrow{\Delta_1^3} & \mathcal{O} \bullet \mathcal{O} \bullet \mathcal{O} & \xrightarrow{=} & \mathcal{O} \bullet \mathcal{O} \bullet \mathcal{O}
\end{array} \tag{10}
$$

□

Example 4.5. The following diagram is commutative (the unlabeled maps are parenthesizations):

$$
\begin{array}{ccccc}
(\mathcal{O} \bullet \mathcal{O}) \bullet \mathcal{O} & \xrightarrow{(\Delta \bullet \mathrm{Id}) \bullet \mathrm{Id}} & \big((\mathcal{O} \bullet \mathcal{O}) \bullet \mathcal{O}\big) \bullet \mathcal{O} & & \\
\downarrow & & \downarrow & \searrow & \\
\mathcal{O} \bullet \mathcal{O} \bullet \mathcal{O} & \xrightarrow{\Delta \bullet \mathrm{Id} \bullet \mathrm{Id}} & (\mathcal{O} \bullet \mathcal{O}) \bullet \mathcal{O} \bullet \mathcal{O} & \longrightarrow & \mathcal{O} \bullet \mathcal{O} \bullet \mathcal{O} \bullet \mathcal{O}
\end{array}
\qquad \mathcal{O} \bullet \mathcal{O} \bullet \mathcal{O} \xrightarrow{\Delta_1^4} \mathcal{O} \bullet \mathcal{O} \bullet \mathcal{O} \bullet \mathcal{O}
$$

Theorem 4.4 has the following corollary:

Corollary 4.6. *Commutativity of the following diagrams are equivalent.*

$$
\begin{array}{ccccc}
 & \overset{\tilde{\Delta}}{\nearrow} & (A \circledcirc A)(\gamma^2 \wr \mathrm{Id}) & \overset{\tilde{\Delta} \wr \mathrm{Id}}{\searrow} & \\
A\gamma^3 & & & & A \circledcirc A \circledcirc A \\
 & \underset{\tilde{\Delta}}{\searrow} & (A \circledcirc A)(\mathrm{Id} \wr \gamma^2) & \underset{\mathrm{Id} \wr \tilde{\Delta}}{\nearrow} &
\end{array} \tag{11}
$$

$$
\begin{array}{ccccccc}
 & \overset{\Delta}{\nearrow} & A \bullet A & \xrightarrow{\Delta \bullet \mathrm{Id}} & (A \bullet A) \bullet A & \searrow & \\
A & & & & & & A \bullet A \bullet A \\
 & \underset{\Delta}{\searrow} & A \bullet A & \xrightarrow[\mathrm{Id} \bullet \Delta]{} & A \bullet (A \bullet A) & \nearrow &
\end{array} \tag{12}
$$

Thus a cooperad could equivalently be defined as $(\mathcal{O}, \Delta, \epsilon)$ where $\mathcal{O}$ is a symmetric sequence, $\Delta : \mathcal{O} \to \mathcal{O} \bullet \mathcal{O}$ so that the analog of Diagram (12) commutes, and Δ is compatible with the counit $\epsilon : \mathcal{O}(1) \to 1_{\otimes}$.

5 Comodules and Coalgebras

Throughout this section, let $(\mathcal{O}, \tilde{\Delta}_{\mathcal{O}}, \tilde{\epsilon})$ be a counital cooperad and M be a symmetric sequence. The definition of cooperad comodules mirrors that of coalgebra comodules (dual to algebra modules). The benefit of viewing cooperads as symmetric sequences is that coalgebras over a cooperad can be viewed, essentially, as a special type of comodule.

5.1 Comodules

Definition 5.1. A left $\mathcal{O}$-comodule is $(M,\, \tilde{\Delta}_M)$ where M is a symmetric sequence and $\tilde{\Delta}_M : M\,\gamma_2 \to \mathcal{O} \circledcirc M$ is compatible with ∂_1^3 and ∂_2^3 and s_0^1.

That is, the following diagrams (analogous to Diagrams (1) and (5)) should commute.

$$
(13)\qquad
\begin{array}{ccccc}
 & \xrightarrow{\tilde{\Delta}_M} & (\mathcal{O} \circledcirc M)\,(\gamma^2 \wr \mathrm{Id}) & \xrightarrow{\tilde{\Delta} \circledcirc \mathrm{Id}} & \\
M\,\gamma^3 & & & & \mathcal{O} \circledcirc \mathcal{O} \circledcirc M \\
 & \xrightarrow{\tilde{\Delta}_M} & (\mathcal{O} \circledcirc M)\,(\mathrm{Id} \wr \gamma^2) & \xrightarrow{\mathrm{Id} \circledcirc \tilde{\Delta}_M} &
\end{array}
$$

$$
(14)\qquad
\begin{array}{ccccccc}
 & M\,\gamma^2 s_0^1 & \xrightarrow{\tilde{\Delta}_M\, s_0^1} & (\mathcal{O} \circledcirc M)\, s_0^1 & \xrightarrow{(\tilde{\epsilon} \circledcirc \mathrm{Id})\, s_0^1} & (\mathbb{1} \circledcirc M)\, s_0^1 & \\
\overset{=}{\nearrow} & & & & & & \overset{=}{\searrow} \\
M & & & \xrightarrow{\qquad\qquad \mathrm{Id} \qquad\qquad} & & & M
\end{array}
$$

As with cooperads, we write Δ_M for the induced universal transformation to the right Kan extension $\Delta_M : M \to \mathcal{O} \bullet M$. There are induced transformations $\tilde{\Delta}_i^{n+1} : \big(\mathcal{O}^{\circledcirc(n-1)} \circledcirc M\big)\, \partial_i^{n+1} \to \mathcal{O}^{\circledcirc n} \circledcirc M$ and $\Delta_i^{n+1} : \mathcal{O}^{\bullet(n-1)} \bullet M \to \mathcal{O}^{\bullet n} \bullet M$.

Theorem 5.2. *Analogous to Theorem 3.18 there is a canonical coaugmented cosimplicial complex as below.*

$$
M \rightleftarrows \mathcal{O} \bullet M \;\substack{\longrightarrow \\ \leftarrow \\ \longrightarrow}\; \mathcal{O}^{\bullet 2} \bullet M \;\substack{\longrightarrow \\ \leftarrow \\ \longrightarrow \\ \leftarrow \\ \longrightarrow}\; \mathcal{O}^{\bullet 3} \bullet M \;\substack{\longrightarrow \\ \leftarrow \\ \longrightarrow \\ \leftarrow \\ \longrightarrow \\ \leftarrow \\ \longrightarrow}\; \cdots
$$

Corollary 5.3. *There are unique transformations $\Delta_M^{[n]} : M \to \mathcal{O}^{\bullet(n-1)} \bullet M$. These are equal to any combination of parenthesization maps and cocomposition maps from their source to their target.*

Remark 5.4. Right $\mathcal{O}$-comodules could be defined similarly. However recent experience suggests that right comodules are most interesting in the non-counital case; in which situation we should use partial cocomposition products rather than cocomposition products. This moves beyond the scope of the current work.

5.2 Coalgebras

Let a be an object of $\mathcal{C}$ and A be a symmetric sequence. Note that a can be viewed as a functor $a : \Sigma_0 \to \mathcal{C}$. Recall the descriptions of the category $\hat{\emptyset}_n$ in Remarks 2.16 and 2.7. We may view $\hat{\emptyset}_n$ either as the category of level n trees with no leaves; or as $\overline{\Sigma}_*^{\wr(n-1)} \subset \Sigma_*^{\wr(n+1)}$, the full subcategory consisting of chains of set maps of the following form.

$$\star \xleftarrow{f_0} S_1 \xleftarrow{f_1} \cdots \xleftarrow{f_{n-2}} S_{n-1} \xleftarrow{f_n} \emptyset$$

Note that the category $\overline{\Sigma}_*^{\wr 0}$ consists of only the trivial chain $(\star \leftarrow \emptyset)$. This is equivalent to Σ_0.

The face and degeneracy maps of $\Sigma_*^{\wr(n+1)}$ induce the following face and degeneracy maps on $\overline{\Sigma}_*^{\wr(n-1)}$. (We introduce an index shift below so that $\bar{\partial}_i^n$ and $\bar{s}_j^n$ map from $\hat{\emptyset}_n = \overline{\Sigma}_*^{\wr(n-1)}$.)

$$\begin{cases} \bar{\partial}_i^n : \overline{\Sigma}_*^{\wr(n-1)} \to \overline{\Sigma}_*^{\wr(n-2)}, & \text{for } 1 \leq i \leq (n-1), \text{ and } n > 1 \\ \bar{s}_i^n : \overline{\Sigma}_*^{\wr(n-1)} \to \overline{\Sigma}_*^{\wr n}, & \text{for } 0 \leq i \leq n \text{ and } n \geq 1 \end{cases}$$

The degeneracy map $\bar{s}_n^n$ doubles $\emptyset$, recognizing that a tree without leaves of level n is also of level $(n+1)$. Note that $\bar{\partial}_1^2 : \overline{\Sigma}_*^{\wr 1} \to \Sigma_0$ coequalizes all chains of face maps from $\overline{\Sigma}_*^{\wr(n-1)}$ to $\overline{\Sigma}_*^{\wr 1}$. We write $\bar{\gamma}^n$ for the composition $\bar{\gamma}^n = (\bar{\partial}_{i_2}^2 \cdots \bar{\partial}_{i_n}^n)$.

Under the identification $\overline{\Sigma}_*^{\wr n} \subset \Sigma_*^{\wr(n+2)}$, Definition 3.2 of symmetric sequence composition restricts to a functor $(A_1 \circledcirc \cdots A_{n-1} \circledcirc a) : \overline{\Sigma}_*^{\wr(n-1)} \to \mathcal{C}$. For example, $A \circledcirc a$ is given by the following.

$$\begin{aligned}\left(\star \xleftarrow{f_0} S \xleftarrow{f_1} \emptyset\right) \longmapsto \;& A(S) \otimes \left(\bigotimes_{s \in S} a(\emptyset)\right) \\ &= A(S) \otimes a^{\otimes |S|}\end{aligned}$$

The right Kan extension of Definition 3.2 restricts to a right Kan extension over $\bar{\gamma}^n : \overline{\Sigma}_*^{\wr(n-1)} \to \Sigma_0$, yielding the following functor.

$$A_1 \bullet \cdots \bullet A_{n-1} \bullet a = \mathrm{R}_{\bar{\gamma}^n}(A_1 \circledcirc \cdots \circledcirc A_{n-1} \circledcirc a) : \Sigma_0 \longrightarrow \mathcal{C}$$

For example, $(A \bullet a) = \prod_{k \geq 0} \Big(A(k) \otimes a^{\otimes k}\Big)^{\Sigma_k}$.

Remark 5.5 (Completed tensor product)**.** In categories with a completed tensor product $\hat{\otimes}$, the completed coendomorphisms $\widehat{\mathrm{COEND}}(A) = \coprod_n \mathrm{Hom}(A, A^{\hat{\otimes} n})$ form a cooperad. Cocomposition is dual to the composition operation on the operad $\mathrm{END}(A^*)$. Dualizing the classical algebra definition, coalgebras should be equivalent to objects equipped with cooperad maps $\widehat{\text{COEND}}(A) \to \mathcal{O}$. The completed tensor should be a right adjoint to Hom so that this is equivalent to maps $A \to \mathcal{O}(n) \hat{\otimes} A^{\hat{\otimes} n}$. In this case, the definition of coalgebras above should use $\hat{\otimes}$ rather than $\otimes$ (though $\otimes$ should still be used for the cooperad). A more detailed survey of this issue is beyond the scope of the current work, which is intended to focus on cooperads.

Definition 5.6. A coalgebra over the cooperad $(\mathcal{O}, \tilde{\Delta}, \tilde{\epsilon})$ is $(c, \tilde{\Delta}_c)$ where c is an object of $\mathcal{C}$ and $\tilde{\Delta}_c : c\,\bar{\gamma}^2 \to \mathcal{O} \circledcirc c$ is compatible with face maps $\bar{\partial}_1^2 = (\gamma^2 \wr \mathrm{Id})$, $\bar{\partial}_2^2 = (\mathrm{Id} \wr \bar{\gamma}^2)$ and degeneracy $\bar{s}_0^1$.

That is, the following diagrams (analogous to Diagrams (13) and (14)) should commute.

$$\begin{array}{ccccc} & \overset{\tilde{\Delta}_c}{\nearrow} & (\mathcal{O} \circledcirc c)(\gamma^2 \wr \mathrm{Id}) & \overset{\tilde{\Delta} \circledcirc \mathrm{Id}}{\searrow} & \\ c\,\bar{\gamma}^3 & & & & \mathcal{O} \circledcirc \mathcal{O} \circledcirc c \\ & \underset{\tilde{\Delta}_c}{\searrow} & (\mathcal{O} \circledcirc c)(\mathrm{Id} \wr \bar{\gamma}^2) & \underset{\mathrm{Id} \circledcirc \tilde{\Delta}_c}{\nearrow} & \end{array} \tag{15}$$

$$\begin{array}{ccccccc} & \overset{=}{\nearrow} & c\,\bar{\gamma}^2 \bar{s}_0^1 \xrightarrow{\tilde{\Delta}_c\, \bar{s}_0^1} (\mathcal{O} \circledcirc c)\, \bar{s}_0^1 \xrightarrow{(\tilde{\epsilon} \circledcirc \mathrm{Id})\, \bar{s}_0^1} (\mathbb{1} \circledcirc c)\, \bar{s}_0^1 & \overset{=}{\searrow} & \\ c & & \xrightarrow{\qquad\qquad\qquad \mathrm{Id} \qquad\qquad\qquad} & & c \end{array} \tag{16}$$

Statements and proofs about left comodules translate into statements and proofs about coalgebras by converting ∂_i^n, s_i^n into $\bar{\partial}_i^n$, $\bar{s}_i^n$. Essentially, coalgebras are left comodules which are concentrated in 0-arity. Write Δ_c for the induced map (in $\mathcal{C}$) $\Delta_c : c \to \mathcal{O} \bullet c$. As with comodules we have $\tilde{\Delta}_i^{n+1} : \left(\mathcal{O}^{\circledcirc(n-1)} \circledcirc c\right) \bar{\partial}_i^{n+1} \to \mathcal{O}^{\circledcirc n} \circledcirc c$ inducing $\Delta_i^{n+1} : \mathcal{O}^{\bullet(n-1)} \bullet c \to \mathcal{O}^{\bullet n} \bullet c$.

Theorem 5.7. *The comultiplication Δ_c defines a canonical coaugmented cosimplicial complex (in $\mathcal{C}$)*

$$c \rightleftarrows \mathcal{O} \bullet c \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \mathcal{O}^{\bullet 2} \bullet c \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \mathcal{O}^{\bullet 3} \bullet c \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \cdots$$

Corollary 5.8. *There are unique $\mathcal{C}$-maps $\Delta^{[n]} : c \longrightarrow \mathcal{O}^{\bullet(n-1)} \bullet c$. These are equal to any combination of parenthesization maps and cocomposition maps from their source to their target.*

6 Examples

We end with a two simple examples of cooperads which are not duals of standard operads. Both of these are constructed via quotient/contraction operations. The (directed) graph cooperad is used in [9] and the contractible Δ complex operad is a generalization.

6.1 The Graph Cooperad

Given a finite set S, a contractible S-graph is a connected, acyclic graph whose vertex set is S. The unoriented graph cooperad has $\overline{\mathrm{GR}}(S)$ equal to the free $\mathbb{Z}$ module generated by all contractible S-graphs. The cocomposition natural transformation $\tilde{\Delta} : \overline{\mathrm{GR}}\,\gamma^2 \to \overline{\mathrm{GR}} \circledcirc \overline{\mathrm{GR}}$ is defined as follows.

Given two graphs G and K, a quotient map of graphs $q : G \twoheadrightarrow K$ is a surjective map from vertices of G onto vertices of K such that $q(v_1, v_2) = \big(q(v_1), q(v_2)\big)$ defines a map sending edges of G to edges and vertices (if $q(v_1) = q(v_2)$) of K, surjecting onto the edges. Note that if $q : G \twoheadrightarrow K$ is a quotient map and v is a vertex of K, then $q^{-1}(v)$ is a subgraph of G. A graph contraction is a quotient map where each $q^{-1}(v)$ is a connected subgraph. Note that there is a bijection between the edges of G and the edges of K union those of the $q^{-1}(v)$.

Suppose G is an S-graph and $f : S \twoheadrightarrow T$ is a surjection of sets. Given $t \in T$, let $\overline{f^{-1}(t)}$ be the maximal subgraph of G supported by the vertices of $f^{-1}(t)$. We say that f induces a graph contraction on G if $\overline{f^{-1}(t)}$ is contractible for each t. In this case, we define the induced contracted graph (G/f) to have vertices T with an edge from vertex t_1 to t_2 if there is an edge in G from the subgraph $\overline{f^{-1}(t_1)}$ to the subgraph $\overline{f^{-1}(t_2)}$.

Cocomposition $\tilde{\Delta}$ takes the element $\big(T \overset{f}{\leftarrow} S\big)$ of $\Sigma_* \wr \Sigma_*$ to the map

$$\overline{\mathrm{GR}}(S) \longrightarrow \overline{\mathrm{GR}}(T) \otimes \Big(\bigotimes_{t\in T} \overline{\mathrm{GR}}(f^{-1}(t))\Big)$$

which takes a S-graph G to $(G/f) \otimes \big(\bigotimes_{t\in T} \overline{f^{-1}(t)}\big)$ if f defines a graph contraction on G, and sends G to 0 otherwise. Since the quotient operation described previously is clearly associative, this defines a symmetric sequence with cocomposition. The counit map sends S-graphs with only one vertex to $1 \in \mathbb{Z}$ and kills all others.

The (directed) graph cooperad is similar to the unoriented graph cooperad. In the category of <u>directed</u>, contractible S-graphs define

$\overrightarrow{\mathrm{GR}}(S) = \overline{\mathrm{GR}}(S)/\sim$, where $\sim$ identifies reversing the orientation of an edge with multiplication of a graph by -1. Cocomposition on $\overline{\mathrm{GR}}$ gives a well-defined map on $\overrightarrow{\mathrm{GR}}$ since reversing an arrow in G will reverse exactly one arrow either in the quotient graph G/f or in one of the $f^{-1}(t)$.

Free nilpotent coalgebras over the graph cooperad can be written as free $\mathbb{Z}$ modules generated by all (finite) graphs with vertices labeled by the primitive elements. The cooperad structure operates by ripping out subgraphs. Explicitly $\tilde{\Delta}$ takes $\big(T \xleftarrow{f} \emptyset\big)$ to the map which takes a graph G to the sum of terms $(G/f) \otimes \big(\bigotimes_{t\in T} \overline{f^{-1}(t)}\big)$ taken over all set maps $f : \mathrm{Vert}(G) \to T$. [With the convention that $G/f = 0$ if f does not define a graph contraction on G.] Note that

$$\tilde{\Delta}(T \xleftarrow{f} \emptyset)$$

will kill a graph G for all sets T with $|T| > |\mathrm{Vert}(G)|$ (because graph contraction maps cannot increase the number of vertices). Thus this coalgebra is nilpotent. We will leave the full proof that this is a free nilpotent coalgebra for the sequel.

The graph cooperad generalizes to the following.

6.2 The CDC Cooperad

By a Δ-complex, we mean what Hatcher [8, Appendix] calls a "singular Δ-complex" or "$s\Delta$-complex". Essentially this is a CW complex whose cells are all (oriented) simplices and whose attaching maps factor through face maps of the simplex. Given a set S, an $S\,\Delta$-complex is a Δ-complex whose 0-cells are labeled by elements of S. The CDC cooperad has $\mathrm{CDC}(S)$ equal to the free $\mathbb{Z}$ module generated by contractible $S\,\Delta$-complexes. Cocomposition is defined similar to that for $\overline{\mathrm{GR}}$.

If T is a subset of the 0-cells of a Δ-complex X, write $\overline{T}$ for the maximal CW subcomplex of X supported by T. Quotient maps for Δ-complexes are CW quotient maps. We say a quotient map $X \twoheadrightarrow Y$ is a contraction if the inverse image of each 0-cell of Y is a contractible subcomplex of X. If X is a $S\,\Delta$-complex then a set surjection $f : S \twoheadrightarrow T$ induces a CW contraction on X if $\overline{f^{-1}(t)}$ is contractible for each $t \in T$. In this case, we define (X/f) to be the quotient of X by the sub CW-complexes $\overline{f^{-1}(t)}$. The cocomposition map of CDC takes $(T \xleftarrow{f} S)$ to the map which sends the $S\,\Delta$-complex X to $(X/f) \otimes \big(\bigotimes_{t\in T} \overline{f^{-1}(t)}\big)$ if f induces a CW contraction on X and 0 otherwise.

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